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ON MACAULAYFICATION OF NOETHERIAN SCHEMES

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Abstract. The Macaulayfication of a Noetherian scheme X is a birational proper morphism from a Cohen-Macaulay scheme to X. In 1978 Faltings gave a Macaulayfication of a quasi-projective scheme if its non-Cohen-Macaulay locus is of dimension 0 or 1. In the present article, we construct a Macaulayfication of Noetherian schemes without any assumption on the non-Cohen-Macaulay locus. Of course, a desingularization is a Macaulayfication and, in 1964, Hironaka already gave a desingularization of an algebraic variety over a field of characteristic 0. Our method, however, to construct a Macaulayfication is independent of the characteristic.

1. Introduction

Let X be a Noetherian scheme. A birational proper morphism $Y \to X$ is said to be a *Macaulayfication* of X if Y is a Cohen-Macaulay scheme. This notion was given by Faltings [9] in analogy with desingularization. Furthermore, he proved that a quasi-projective scheme over a Noetherian ring A has a Macaulayfication if its non-Cohen-Macaulay locus is of dimension 0 or 1 and A possesses a dualizing complex. We should mention that his method of constructing a Macaulayfication is independent of the characteristic of the ring A unlike Hironaka's desingularization [19].

Furthermore, several authors are interested in Macaulayfication. For example, Goto [10] gave a Macaulayfication of the spectrum of certain Buchsbaum rings. We can give a Macaulayfication of the spectrum of arbitrary Buchsbaum rings by Faltings' method. Goto's Macaulayfication, however, is better than Faltings' one in a sense. Indeed, Goto's one is a finite morphism though Faltings' one is not. Furthermore, Goto [11] and Schenzel [29], independently, refined Faltings' Macaulayfication in the case that the non-Cohen-Macaulay locus is of dimension 0.

On the other hand, Brodmann [4], [5] gave other Macaulayfications under the same assumption as Faltings. His method is quite different from Faltings' one. Indeed, his Macaulayfication preserves normality and regularity. He also showed in [6] that there exists no minimal Macaulayfication even if X is a surface.

Recently, the author [21] gave a Macaulayfication of quasi-projective schemes whose non-Cohen-Macaulay locus is of dimension 2. In the present article, we improve it. The main theorem of this article is as follows.

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Theorem 1.1. *Let A be a Noetherian ring possessing a dualizing complex and X a separated, of finite type scheme over* $\operatorname{Spec} A$. *Then X has a Macaulayfication.*

To make a survey, we sketch the construction of our Macaulayfication. First we consider the case of affine schemes. Let A be a Noetherian local ring possessing a dualizing complex. Then the non-Cohen-Macaulay locus V of $X = \operatorname{Spec} A$ is closed. We assume that A is equidimensional and fix an integer s such that $\dim V \leq s < \dim X$. In this case our Macaulayfication of X consists of successive $s+1$ blowing-ups

$$Y_{s,1} \to Y_{s,2} \to \cdots \to Y_{s,s} \to Y_{s,s+1} \to X$$

such that

$$\operatorname{depth} \mathcal{O}_{Y_{s,i},p} \geq d - i + 1 \quad \text{for all closed points } p \text{ on } Y_{s,i}.$$

The schemes $Y_{s,s}$ and $Y_{s,s+1}$ were essentially given by Faltings. Furthermore, if $s > 0$, then we can give another Macaulayfication of X consisting of a finite morphism $h_{s,1}$ and successive s blowing-ups

$$Z_{s,1} \xrightarrow{h_{s,1}} Y_{s,2} \to \cdots \to Y_{s,s} \to Y_{s,s+1} \to X.$$

Since s is arbitrary, we have distinct $2(\dim X - \dim V)$ Macaulayfications of X if $\dim V > 0$. In Section 4, we give the precise statement and its proof. We compute an example of a Macaulayfication in Appendix B

The center of each blowing-up is the ideal generated by a subsystem of certain system of parameters for A, named a *p-standard system of parameters* by Cuong [7]. In particular, the center of the first blowing-up is the ideal generated by an unconditioned strong d-sequence (for short d^+-sequence), which was studied by Goto and Yamagishi. We discuss a p-standard system of parameters in Sections 2 and 3. Since the theory of d^+-sequences plays a key role, this article includes proofs of a few results of [15] in Appendix A.

In Section 5, we prove Theorem 1.1 by using the result in the preceding section. It is a routine established by Faltings.

In Section 6, we give an application of Macaulayfication. A dualizing complex is an important notion for commutative algebra and algebraic geometry. It is well-known that a homomorphic image of a finite-dimensional Gorenstein ring has a dualizing complex. Indeed if B is a finite-dimensional Gorenstein ring, then its injective resolution $I^\bullet$ is a dualizing complex of B. If A is a homomorphic image of B, then $\operatorname{Hom}_B(A, I^\bullet)$ is a dualizing complex of A. In 1979 Sharp [30] posed a conjecture: the converse is true. Aoyama and Goto [1], [2] gave a partial answer to Sharp's conjecture by using Faltings' Macaulayfication. They showed that Sharp's conjecture is true for a Noetherian local ring whose non-Cohen-Macaulay locus is of dimension 0 or 1. Their argument still works in general. We show the following theorem.

Theorem 1.2. *A Noetherian ring A possessing a dualizing complex is a homomorphic image of a finite-dimensional Gorenstein ring if A satisfies one of the following three conditions*:

1. *A has no embedded prime ideal and $t(\mathfrak{p}) - \operatorname{ht} \mathfrak{p}$ is locally constant on* $\operatorname{Spec} A$ *where t denotes the codimension function of A*;
2. *A is an integral domain*;
3. *A is a local ring.*

Throughout this article except for Sections 5 and 6, A denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$. For a graded ring $R = \bigoplus_{n\geq 0} R_n$, R_+ denotes the irrelevant ideal $\bigoplus_{n>0} R_n$. We refer the reader to [17], [18] and [22] for other unexplained terminology.

2. A p-standard system of parameters, I

In this section, we state the definition and properties of a p-standard system of parameters, which was introduced by Cuong [7].

First we recall definitions of d-sequences and unconditioned strong d-sequences. They were given by Huneke [20] and GotoYamagishi [15], respectively, but our definition of d-sequences is slightly different from Huneke's original one.

Definition 2.1. Let M be an A-module. A sequence $x_1, \dots, x_d$ of elements in A is said to be a *d-sequence* on M if

$$(x_1, \dots, x_{i-1})M \colon x_i x_j = (x_1, \dots, x_{i-1})M \colon x_j \quad \text{for any } 1 \leq i \leq j \leq d.$$

A sequence $x_1, \dots, x_d$ of elements in A is said to be an unconditioned strong d-sequence (for short, a *d^+-sequence*) on M if $x_1^{n_1}, \dots, x_d^{n_d}$ is a d-sequence on M for any positive integers $n_1, \dots, n_d$ and in any order.

The following lemma was first given by Goto and Shimoda [12, Lemma 4.2] for the system of parameters for a Buchsbaum ring, which is a typical example of d-sequences. Since we often use it, we give a proof.

Lemma 2.2 ([15, Theorem 1.3])**.** *Let M be an A-module and $x_1, \dots, x_d$ a d-sequence on M. If we put $\mathfrak{q} = (x_1, \dots, x_d)$, then*

$$(x_1, \dots, x_{i-1})M \colon x_i \cap \mathfrak{q}^n M = (x_1, \dots, x_{i-1})\mathfrak{q}^{n-1} M \tag{2.2.1}$$

for all $n > 0$ and $1 \leq i \leq d+1$, where we set $x_{d+1} = 1$ and $(x_1, \dots, x_{i-1}) = (0)$ if $i = 1$.

Proof. We work by induction on n and i. If $i = d+1$, then it is trivial. Let $i \leq d$ and assume that

$$(x_1, \dots, x_i)M \colon x_{i+1} \cap \mathfrak{q}^n M = (x_1, \dots, x_i)\mathfrak{q}^{n-1} M.$$

If a is in the left hand side of (2.2.1), then

$$\begin{aligned} a &\in (x_1, \dots, x_{i-1})M \colon x_i \cap \mathfrak{q}^n M \\ &\subseteq (x_1, \dots, x_{i-1})M \colon x_i x_{i+1} \cap \mathfrak{q}^n M \\ &= (x_1, \dots, x_{i-1})M \colon x_{i+1} \cap \mathfrak{q}^n M \\ &\subseteq (x_1, \dots, x_i)M \colon x_{i+1} \cap \mathfrak{q}^n M \\ &= (x_1, \dots, x_i)\mathfrak{q}^{n-1} M. \end{aligned}$$

Let $a = b + x_i c$ with $b \in (x_1, \dots, x_{i-1})\mathfrak{q}^{n-1} M$ and $c \in \mathfrak{q}^{n-1} M$. Then

$$c \in (x_1, \dots, x_{i-1})M \colon x_i^2 = (x_1, \dots, x_{i-1})M \colon x_i.$$

If $n = 1$, then $a = b + x_i c \in (x_1, \dots, x_{i-1})M$.

If $n > 1$, then

$$\begin{aligned} c &\in (x_1, \dots, x_{i-1})M \colon x_i \cap \mathfrak{q}^{n-1} M \\ &= (x_1, \dots, x_{i-1})\mathfrak{q}^{n-2} M \end{aligned}$$

by the induction hypothesis. Thus $a = b + x_i c \in (x_1, \dots, x_{i-1})\mathfrak{q}^{n-1} M$. ∎

To state the definition of a p-standard system of parameters, we need the following definition and lemmas. They were given by Schenzel [26] [27] and [28].

Definition 2.3. Let M be a finitely generated A-module. An ideal $\mathfrak{a}(M)$ is defined to be

$$\mathfrak{a}(M) = \prod_{0 \le i < \dim M} \operatorname{ann} H^i_{\mathfrak{m}}(M)$$

where $\operatorname{ann} H^i_{\mathfrak{m}}(M)$ denotes the annihilator of $H^i_{\mathfrak{m}}(M)$.

If A is a homomorphic image of a Gorenstein local ring B, then the local duality theorem says that $\operatorname{ann} H^i_{\mathfrak{m}}(M) = \operatorname{ann} \operatorname{Ext}^{n-i}_B(M, B)$, where $n = \dim B$. Therefore

$$\mathfrak{a}(M) = \prod_{i=n-\dim M+1}^{n} \operatorname{ann} \operatorname{Ext}^i_B(M, B).$$

Of course, M is Cohen-Macaulay if and only if $\mathfrak{a}(M) = A$.

Lemma 2.4. *Let M be a finitely generated A-module. If A possesses a dualizing complex, then the following statements hold*:

1. $\dim A/\mathfrak{a}(M) < \dim M$.
2. *Let $\mathfrak{p}$ be a prime ideal of A. Then $\mathfrak{a}(M) \not\subseteq \mathfrak{p}$ if and only if $M_{\mathfrak{p}}$ is a Cohen-Macaulay $A_{\mathfrak{p}}$-module and* $\dim A/\mathfrak{p} + \dim M_{\mathfrak{p}} = \dim M$.
3. *If M is equidimensional, then the non-Cohen-Macaulay locus of M coincides with $V(\mathfrak{a}(M))$.*

Proof. (1): See [28, p. 46]. (2): See [28, Satz 2.4.6]. (3): See [28, p. 52]. ☐

Lemma 2.5. *Let M be a finitely generated A-module of dimension $d > 0$ and $x_1, \dots, x_d$ a system of parameters for M. Then*

$$(x_1, \dots, x_{i-1})M : x_i \subseteq (x_1, \dots, x_{i-1})M : \mathfrak{a}(M) \quad \textit{for all } 1 \le i \le d.$$

In particular, if $x_i \in \mathfrak{a}(M)$, then the equality holds.

Proof. See [27, Theorem 3] or [28, Satz 2.4.2]. ☐

Lemma 2.5 is always true but Lemma 2.4 is false without dualizing complex. We find a counterexample in [26].

Now we give the definition of a p-standard system of parameters, but it is slightly different from Cuong's original one.

Definition 2.6. Let M be a finitely generated A-module of dimension $d > 0$, $x_1, \dots, x_d$ a system of parameters for M and s an integer such that $0 \le s < d$. We say that $x_1, \dots, x_d$ is a *p-standard system of parameters* of type s if

1. $x_{s+1}, \dots, x_d \in \mathfrak{a}(M)$;
2. $x_i \in \mathfrak{a}(M/(x_{i+1}, \dots, x_d)M)$ for all $1 \le i \le s$.

In the case of $s = d - 1$, our definition coincides with Cuong's one.

As a consequence of Lemma 2.4, we can find a p-standard system of parameters for any finitely generated module.

Theorem 2.7. *Let M be a finitely generated A-module of dimension $d > 0$ and s an integer such that $\dim A/\mathfrak{a}(M) \le s < d$. If A has a dualizing complex, then there exists a p-standard system of parameters of type s for M.*

Proof. Since $d - \dim A/\mathfrak{a}(M) \geq d - s$, there exist $d-s$ elements $x_{s+1}, \ldots, x_d \in \mathfrak{a}(M)$ such that $\dim M/(x_{s+1}, \ldots, x_d)M = d - (d-s) = s$.

If a subsystem of parameters $x_{i+1}, \ldots, x_d$ for M is given, then there exists $x_i \in \mathfrak{a}(M/(x_{s+1}, \ldots, x_d)M)$ such that $\dim M/(x_i, \ldots, x_d)M = i - 1$, because $\dim A/\mathfrak{a}(M/(x_{i+1}, \ldots, x_d)M) < i$. □

The following proposition comes from Lemma 2.5 at once.

Proposition 2.8. *Let M be a finitely generated A-module of dimension $d > 0$ and $x_1, \ldots, x_d$ a p-standard system of parameters of type s for M. Then the sequence $x_{s+1}, \ldots, x_d$ is a d^+-sequence on $M/(y_1, \ldots, y_u)M$ for any subsystem of parameters $y_1, \ldots, y_u$ for $M/(x_{s+1}, \ldots, x_d)M$. In particular, $x_{s+1}, \ldots, x_d$ is a d^+-sequence on M itself.*

Proof. Since $y_1, \ldots, y_u, x_{s+1}^{n_{s+1}}, \ldots, x_i^{n_i}$ and $y_1, \ldots, y_u, x_{s+1}^{n_{s+1}}, \ldots, x_{i-1}^{n_{i-1}}, x_i^{n_i} x_j^{n_j}$ are two subsystems of parameters for M,

$$\begin{aligned}(y_1, \ldots, y_u, x_{s+1}^{n_{s+1}}, \ldots, x_{i-1}^{n_{i-1}})M \colon x_i^{n_i} &= (y_1, \ldots, y_u, x_{s+1}^{n_{s+1}}, \ldots, x_{i-1}^{n_{i-1}})M \colon \mathfrak{a}(M) \\ &= (y_1, \ldots, y_u, x_{s+1}^{n_{s+1}}, \ldots, x_{i-1}^{n_{i-1}})M \colon x_i^{n_i} x_j^{n_j}\end{aligned}$$

for any integers $n_{s+1}, \ldots, x_d$ and $s + 1 \leq i \leq j \leq d$. □

The main theorem of this section is as follows.

Theorem 2.9. *Let M be a finitely generated A-module of dimension $d > 0$, $x_1, \ldots, x_d$ a p-standard system of parameters of type s for M and $y_1, \ldots, y_u$ a subsystem of parameters for $M/(x_i, \ldots, x_d)M$ where $2 \leq i \leq d$ and $1 \leq u < i$. If $y_u \in \mathfrak{a}(M)$ or $y_u \in \mathfrak{a}(M/(x_i, \ldots, x_d)M)$, then*

$$(y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v y_u = (y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_u \tag{2.9.1}$$

for any $1 \leq v \leq u$ and $\Lambda \subseteq \{i, \ldots, d\}$. In particular, by letting $\Lambda = \emptyset$, we have

$$(y_1, \ldots, y_{v-1})M \colon y_v y_u = (y_1, \ldots, y_{v-1})M \colon y_u. \tag{2.9.2}$$

Here $(x_\lambda \mid \lambda \in \Lambda)$ denotes the ideal generated by $\{x_\lambda \mid \lambda \in \Lambda\}$ and we put

$$(y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda) = (y_1, \ldots, y_{v-1}) + (x_\lambda \mid \lambda \in \Lambda).$$

Proof. If $y_u \in \mathfrak{a}(M)$, then the both side of (2.9.1) coincide with

$$(y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon \mathfrak{a}(M)$$

because of Lemma 2.5.

In the case of $y_u \in \mathfrak{a}(M/(x_i, \ldots, x_d)M)$, we work by induction on the number of elements in Λ. If $\Lambda = \{i, \ldots, d\}$, then (2.9.1) comes from Lemma 2.5 applied to $M/(x_i, \ldots, x_d)M$.

Assume that $\Lambda \neq \{i, \ldots, d\}$ and let l be the largest element of $\{i, \ldots, d\} \setminus \Lambda$. Let a be an element of the left hand side of (2.9.1). Then we have

$$\begin{aligned}a &\in (y_1, \ldots, y_{v-1}, x_l, x_\lambda \mid \lambda \in \Lambda)M \colon y_v y_u \\ &= (y_1, \ldots, y_{v-1}, x_l, x_\lambda \mid \lambda \in \Lambda)M \colon y_u\end{aligned}$$

by the induction hypothesis. We put $y_u a = x_l b + c$ with

$$c \in (y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M.$$

Then we obtain that

$$\begin{aligned} b &\in (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M : y_v x_l \\ &= (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M : x_l. \end{aligned}$$

Indeed, if $l \leq s$, then $x_l \in \mathfrak{a}(M/(x_{l+1}, \dots, x_d)M)$ and $l+1, \dots, d \in \Lambda$. Hence we can apply Lemma 2.5 to $M/(x_{l+1}, \dots, x_d)M$. On the other hand, if $l > s$, then $x_l \in \mathfrak{a}(M)$. We can apply Lemma 2.5 to M itself. Therefore

$$y_u a = x_l b + c \in (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M.$$

That is, $a \in (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M : y_u$. The proof is completed. □

We use the following corollary for our Macaulayfication in Section 4.

Corollary 2.10. *Let M be a finitely generated A-module of dimension $d > 0$, $x_1, \dots, x_d$ a p-standard system of parameters of type s for M and $y_1, \dots, y_u$ a subsystem of parameters for $M/(x_i, \dots, x_d)M$ where $1 \leq i \leq d$ and $1 \leq u < i$. Then the sequence $x_i, \dots, x_d$ is a d-sequence on $M/(y_1, \dots, y_u)M$. In particular, $x_1, \dots, x_d$ is a d-sequence on M itself.*

Proof. Let $i \leq j \leq k \leq d$. By applying (2.9.2) to the subsystem of parameters $y_1, \dots, y_u, x_i, \dots, x_k$ for $M/(x_{k+1}, \dots, x_d)M$, we obtain

$$(y_1, \dots, y_u, x_i, \dots, x_{j-1})M : x_j x_k = (y_1, \dots, y_u, x_i, \dots, x_{j-1})M : x_k. \qquad \square$$

3. A p-standard system of parameters, II

This section is devoted to the proof of the following theorem.

Theorem 3.1. *Let M be a finitely generated A-module of dimension $d > 0$ and $x_1, \dots, x_d$ a p-standard system of parameters of type s for M. We put $\mathfrak{q}_i = (x_i, \dots, x_d)$ for all $1 \leq i \leq d$. Then for any positive integers $1 \leq i \leq j \leq d$ and $n_i, \dots, n_j$, we obtain the following statements:*

(A_{ij}) *If $y_1, \dots, y_u$ is a subsystem of parameters for $M/\mathfrak{q}_i M$, then*

(3.1.1)
$$\begin{aligned} (y_1, \dots, y_u, x_k, \dots, x_{l-1})M : x_l \cap [(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \\ = (y_1, \dots, y_u)M + (x_k, \dots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k - 1} \cdots \mathfrak{q}_j^{n_j} M \end{aligned}$$

for arbitrary integers k and l such that $i \leq k \leq j$ and $k \leq l \leq d+1$. Here we set $x_{d+1} = 1$.

(B_{ij}) *If $y_1, \dots, y_u$ is a subsystem of parameters for $M/\mathfrak{q}_i M$, then*

(3.1.2)
$$\begin{aligned} &[(y_1, \dots, y_{u-1})M + (x_k, \dots, x_l)\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : y_u \\ &\quad = (x_k, \dots, x_l)\{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : y_u\} \\ &\qquad + (y_1, \dots, y_{u-1})M : y_u \end{aligned}$$

for arbitrary integers k and l such that $i \leq k \leq j$ and $k \leq l \leq d$. In particular, by letting $l = d$, we have

$$\begin{aligned} &[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k+1} \cdots \mathfrak{q}_j^{n_j} M] : y_u \\ &\quad = \mathfrak{q}_k\{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] : y_u\} \\ &\qquad + (y_1, \dots, y_{u-1})M : y_u \end{aligned}$$

for all $i \leq k \leq j$.

(C_{ij}) *If* $y_1, \ldots, y_u$ *is a subsystem of parameters for* $M/\mathfrak{q}_i M$, *then*

$$[(y_1, \ldots, y_{u-1})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \colon y_u \subseteq (y_1, \ldots, y_{u-1})M \colon y_u + \mathfrak{q}_i^{n_i - 1} \cdots \mathfrak{q}_j^{n_j} M. \tag{3.1.3}$$

(D_{ij}) *If* $y_1, \ldots, y_u$ *is a subsystem of parameters for* $M/\mathfrak{q}_i M$, *then*

$$[(y_1, \ldots, y_{u-1})M + \mathfrak{q}_i \cdots \mathfrak{q}_j M] \colon y_u \cap x_i M \subseteq x_i \{[(y_1, \ldots, y_{u-1})M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_u\} + (y_1, \ldots, y_{u-1})M. \tag{3.1.4}$$

Here we put $\mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M = M$ *if* $i = j$.

(E_{ij}) *Let* k *be an integer such that* $2 \leq k \leq i$. *If* $y_1, \ldots, y_u$ *is a subsystem of parameters for* $M/\mathfrak{q}_k M$ *such that* $y_u \in \mathfrak{a}(M/\mathfrak{q}_k M)$, *then*

$$[(y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \colon y_v y_u = [(y_1, \ldots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \colon y_u \tag{3.1.5}$$

for any $1 \leq v \leq u$ *and* $\Lambda \subseteq \{k, \ldots, i-1\}$.

We divide the proof into several steps. Roughly speaking, we work by induction on $k = j - i$.

DIAGRAM OF THE INDUCTION

$\forall i \colon (A_{ii}), (B_{ii}), \ldots, (E_{i,i+k-1})$	$\Rightarrow \forall i \colon (A_{i,i+k})$	(Steps 1 and 5);
$\forall i \colon (A_{ii}), (B_{ii}), \ldots, (E_{i,i+k-1}), (A_{i,i+k})$	$\Rightarrow \forall i \colon (B_{i,i+k})$	(Step 2);
$\forall i \colon (A_{ii}), (B_{ii}), \ldots, (A_{i,i+k}), (B_{i,i+k})$	$\Rightarrow \forall i \colon (C_{i,i+k})$	(Steps 3 and 6);
$\forall i \colon (A_{ii}), (B_{ii}), \ldots, (B_{i,i+k}), (C_{i,i+k})$	$\Rightarrow \forall i \colon (D_{i,i+k})$	(Steps 4 and 7);
$\forall i \colon (A_{ii}), (B_{ii}), \ldots, (C_{i,i+k}), (D_{i,i+k})$	$\Rightarrow \forall i \colon (E_{i,i+k})$	(Steps 3, 8 and 9).

Step 1. (A_{ii}) *is true.*

Proof. Corollary 2.10 says that $x_i, \ldots, x_d$ is a d-sequence on $M/(y_1, \ldots, y_u)M$. Therefore (A_{ii}) coincides with Lemma 2.2. □

Step 2. *If* $j \geq i$, *then* (B_{ij}) *comes from* (A_{ij}).

Proof. Let a be an element of the left hand side of (3.1.2) and put $y_u a = x_l b + c$ with $b \in \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M$ and $c \in (y_1, \ldots, y_{u-1})M + (x_k, \ldots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M$. By using (A_{ij}), we obtain

$$\begin{aligned} b &\in (y_1, \ldots, y_u, x_k, \ldots, x_{l-1})M \colon x_l \cap \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M \\ &\subseteq (y_1, \ldots, y_u)M + (x_k, \ldots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k - 1} \cdots \mathfrak{q}_j^{n_j} M. \end{aligned}$$

Let $b = y_u a' + c'$ with

$$c' \in (y_1, \ldots, y_{u-1})M + (x_k, \ldots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k - 1} \cdots \mathfrak{q}_j^{n_j} M.$$

Then

$$a' \in [(y_1, \ldots, y_{u-1})M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \colon y_u$$

and

$$a - x_l a' \in [(y_1, \ldots, y_{u-1})M + (x_k, \ldots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \colon y_u.$$

By induction on l, we find that a is in the right hand side of (3.1.2). The opposite inclusion is obvious. □

Step 3. *(B_{ii}) implies (C_{ii}) and (E_{ii}).*

Proof. By using (B_{ii}), we have

$$\begin{aligned}[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^{n_i} M] \colon y_u \\ = \mathfrak{q}_i^{n_i - 1}\{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_i M] \colon y_u\} \\ + (y_1, \dots, y_{u-1})M \colon y_u\end{aligned}$$

and

$$\begin{aligned}[(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} M] \colon y_v y_u \\ = \mathfrak{q}_i^{n_i - 1}\{[(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i M] \colon y_v y_u\} \\ + (y_1, \dots, y_{v-1})M \colon y_v y_u.\end{aligned}$$

Furthermore, if $y_u \in \mathfrak{a}(M/\mathfrak{q}_k M)$, then (2.9.2) says that

$$(y_1, \dots, y_{v-1})M \colon y_v y_u = (y_1, \dots, y_{v-1})M \colon y_u.$$

Thus we may assume that $n_i = 1$. Then (C_{ii}) is trivial and (E_{ii}) is included in Theorem 2.9. □

Step 4. *(D_{ii}) is trivial.*

Proof. Indeed, the right hand side of (3.1.4) is $(y_1, \dots, y_{u-1}, x_i)M$. □

Step 5. *If $j > i$, then (A_{ij}) comes from $(A_{i+1,j})$ and $(C_{i+1,j})$.*

Proof. We first show that

$$\begin{aligned}(3.1.6) \quad (y_1, \dots, y_u, x_i, \dots, x_{l-1})M \colon x_l \cap [(y_1, \dots, y_{u-1})M + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] \\ \subseteq (y_1, \dots, y_u)M + (x_i, \dots, x_{l-1})\mathfrak{q}_{i+1}^{n_{i+1}-1} \cdots \mathfrak{q}_j^{n_j} M\end{aligned}$$

for all $i \le l \le d+1$. If $l = d+1$, then (3.1.6) is trivial. Assume that $l \le d$. Let a be an element of the left hand side of (3.1.6). By applying $(A_{i+1,j})$ to a subsystem of parameters $y_1, \dots, y_u, x_i$ for $M/\mathfrak{q}_{i+1}M$, we obtain

$$\begin{aligned}a \in (y_1, \dots, y_u, x_i, \dots, x_{l-1})M \colon x_l \cap [(y_1, \dots, y_u, x_i)M + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] \\ = (y_1, \dots, y_u, x_i)M + (x_{i+1}, \dots, x_{l-1})\mathfrak{q}_{i+1}^{n_{i+1}-1} \cdots \mathfrak{q}_j^{n_j} M.\end{aligned}$$

If we put $a = x_i b + c$ with

$$c \in (y_1, \dots, y_u)M + (x_{i+1}, \dots, x_{l-1})\mathfrak{q}_{i+1}^{n_{i+1}-1} \cdots \mathfrak{q}_j^{n_j} M,$$

then

$$\begin{aligned}b \in [(y_1, \dots, y_u)M + \mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] \colon x_i \\ \subseteq (y_1, \dots, y_u)M \colon x_i + \mathfrak{q}_{i+1}^{n_{i+1}-1} \cdots \mathfrak{q}_j^{n_j} M\end{aligned}$$

by $(C_{i+1,j})$ applied to a subsystem of parameters $y_1, \dots, y_u, x_i$ for $M/\mathfrak{q}_{i+1}M$. If $l = i$, then the left hand side of (3.1.6) is contained in

$$(y_1, \dots, y_u)M \colon x_i \cap (y_1, \dots, y_u, x_i, \dots, x_d)M = (y_1, \dots, y_u)M.$$

Therefore a is in the right hand side.

Next we show (3.1.1). In the case of $k = i$, we work by induction on l and n_i. Let a be an element of the left hand side of (3.1.1). If $l = d+1$, then there is nothing to prove. Assume that $i+1 \le l \le d$. Since $x_i, \dots, x_d$ is a d-sequence on $M/(y_1, \dots, y_u)M$,

$$(y_1, \dots, y_u, x_i, \dots, x_{l-1})M \colon x_l \subseteq (y_1, \dots, y_u, x_i, \dots, x_l)M \colon x_{l+1}$$

as the proof of Lemma 2.2. Hence we have

$$\begin{aligned} a &\in (y_1, \dots, y_u, x_i, \dots, x_l)M \colon x_{l+1} \cap [(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \\ &= (y_1, \dots, y_u)M + (x_i, \dots, x_l)\mathfrak{q}_i^{n_i-1} \cdots \mathfrak{q}_j^{n_j} M \end{aligned}$$

by the induction hypothesis. Let $a = x_l a' + b$ with $a' \in \mathfrak{q}_i^{n_i-1} \cdots \mathfrak{q}_j^{n_j} M$ and

$$b \in (y_1, \dots, y_u)M + (x_i, \dots, x_{l-1})\mathfrak{q}_i^{n_i-1} \cdots \mathfrak{q}_j^{n_j} M.$$

Since $x_i, \dots, x_d$ is a d-sequence on $M/(y_1, \dots, y_u)M$,

$$a' \in (y_1, \dots, y_u, x_i, \dots, x_{l-1})M \colon x_l^2 = (y_1, \dots, y_u, x_i, \dots, x_{l-1})M \colon x_l.$$

Therefore we get

$$\begin{aligned} a' &\in (y_1, \dots, y_u, x_i, \dots, x_{l-1})M \colon x_l \cap \mathfrak{q}_i^{n_i-1} \dots \mathfrak{q}_j^{n_j} M \\ &\subseteq \begin{cases} (y_1, \dots, y_u)M + (x_i, \dots, x_{l-1})\mathfrak{q}_{i+1}^{n_{i+1}-1} \cdots \mathfrak{q}_j^{n_j} M & \text{if } n_i = 1; \\ (y_1, \dots, y_u)M + (x_i, \dots, x_{l-1})\mathfrak{q}_i^{n_i-2} \dots \mathfrak{q}_j^{n_j} M & \text{if } n_i > 1, \end{cases} \end{aligned}$$

by using (3.1.6) or the induction hypothesis, respectively. Since $x_l \in \mathfrak{q}_{i+1} \subseteq \mathfrak{q}_i$, $a = x_l a' + b$ is in the right hand side of (3.1.1). If $l = i$, then

$$\begin{aligned} a &\in (y_1, \dots, y_u)M \colon x_i \cap [(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \\ &\subseteq (y_1, \dots, y_u)M \colon x_i \cap (y_1, \dots, y_u, x_i, \dots, x_d)M \\ &= (y_1, \dots, y_u)M \end{aligned}$$

because of Lemma 2.2. The proof is completed if $k = i$.

In the case of $k > i$, we work by induction on n_i. Assume that $l \le d$. Let a be an element of the left hand side of (3.1.1). Since $(x_i) + \mathfrak{q}_i^{n_i} = (x_i) + \mathfrak{q}_{i+1}^{n_i}$, we have

$$\begin{aligned} a &\in (y_1, \dots, y_u, x_i, x_k, \dots, x_{l-1})M \colon x_l \cap [(y_1, \dots, y_u, x_i)M + \mathfrak{q}_{i+1}^{n_i+n_{i+1}} \cdots \mathfrak{q}_j^{n_j} M] \\ &= \begin{cases} (y_1, \dots, y_u, x_i)M + (x_{i+1}, \dots, x_{l-1})\mathfrak{q}_{i+1}^{n_i+n_{i+1}-1} \cdots \mathfrak{q}_j^{n_j} M & \text{if } k = i+1; \\ (y_1, \dots, y_u, x_i)M + (x_k, \dots, x_{l-1})\mathfrak{q}_{i+1}^{n_i+n_{i+1}} \cdots \mathfrak{q}_k^{n_k-1} \cdots \mathfrak{q}_j^{n_j} M & \text{if } k > i+1 \end{cases} \\ &= (y_1, \dots, y_u, x_i)M + (x_k, \dots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k-1} \cdots \mathfrak{q}_j^{n_j} M. \end{aligned}$$

Here we applied $(A_{i+1,j})$ to the subsystem of parameters $y_1, \dots, y_u, x_i$ for $M/\mathfrak{q}_{i+1}M$. Taking intersection with $(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M$,

$$\begin{aligned} a &\in (y_1, \dots, y_u, x_i)M \cap [(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \\ &\quad + (x_k, \dots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k-1} \cdots \mathfrak{q}_j^{n_j} M \\ &= (y_1, \dots, y_u)M + x_i \mathfrak{q}_i^{n_i-1} \cdots \mathfrak{q}_j^{n_j} M \\ &\quad + (x_k, \dots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k-1} \cdots \mathfrak{q}_j^{n_j} M. \end{aligned}$$

Here we used (3.1.1) in the case of $k = i$ to show the equation. Let $a = x_i a' + b$ with $a' \in \mathfrak{q}_i^{n_i-1} \cdots \mathfrak{q}_j^{n_j} M$ and

$$b \in (y_1, \dots, y_u)M + (x_k, \dots, x_{l-1})\mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_k^{n_k-1} \cdots \mathfrak{q}_j^{n_j} M.$$

By applying (2.9.2) to the subsystem of parameters $y_1, \dots, y_u, x_k, \dots, x_{l-1}, x_i, x_l$ for $M/\mathfrak{q}_{l+1}M$, we obtain

$$a' \in (y_1, \dots, y_u, x_k, \dots, x_{l-1})M \colon x_i x_l = (y_1, \dots, y_u, x_k, \dots, x_{l-1})M \colon x_l.$$

Therefore we have

$$\begin{aligned} a' &\in (y_1, \dots, y_u, x_k, \dots, x_{l-1})M \colon x_l \cap \mathfrak{q}_i^{n_i-1} \cdots \mathfrak{q}_j^{n_j} M \\ &\subseteq \begin{cases} (y_1, \dots, y_u)M + (x_k, \dots, x_{l-1})\mathfrak{q}_{i+1}^{n_{i+1}} \cdots \mathfrak{q}_k^{n_k-1} \cdots \mathfrak{q}_j^{n_j} M & \text{if } n_i = 1; \\ (y_1, \dots, y_u)M + (x_k, \dots, x_{l-1})\mathfrak{q}_i^{n_i-1} \cdots \mathfrak{q}_k^{n_k-1} \cdots \mathfrak{q}_j^{n_j} M & \text{if } n_i > 1 \end{cases} \end{aligned}$$

by $(A_{i+1,j})$ or the induction hypothesis, respectively. Therefore $a = x_i a' + b$ is in the right hand side of (3.1.1). The opposite inclusion is obvious. □

Step 6. *If $j > i$, then (C_{ij}) comes from (B_{ij}), (C_{ii}), $(C_{i+1,j})$ and $(E_{i+1,j})$.*

Proof. We first show that

(3.1.7)
$$(y_1, \dots, y_{u-1}, x_i)M \colon y_u \cap (y_1, \dots, y_{u-1}, x_i, \dots, x_l)M = (y_1, \dots, y_{u-1}, x_i)M$$

for all $i \le l \le d$. We work by induction on l. If $l = i$, then there is nothing to prove. Assume that $l > i$ and let a be an element of the left hand side of (3.1.7). We put $a = x_l b + c$ with $c \in (y_1, \dots, y_{u-1}, x_i, \dots, x_{l-1})M$. By applying (2.9.2) to a subsystem of parameters $y_1, \dots, y_{u-1}, x_i, \dots, x_{l-1}, y_u, x_l$ for $M/\mathfrak{q}_{l+1}M$, we have

$$\begin{aligned} b &\in (y_1, \dots, y_{u-1}, x_i, \dots, x_{l-1})M \colon y_u x_l \\ &= (y_1, \dots, y_{u-1}, x_i, \dots, x_{l-1})M \colon x_l. \end{aligned}$$

Thus we get

$$\begin{aligned} a &\in (y_1, \dots, y_{u-1}, x_i)M \colon y_u \cap (y_1, \dots, y_{u-1}, x_i, \dots, x_{l-1})M \\ &= (y_1, \dots, y_{u-1}, x_i)M \end{aligned}$$

by the induction hypothesis. The opposite inclusion is obvious.

Next we show (3.1.3). In the same way as Step 3, we may assume that $n_i = \cdots = n_j = 1$. Let a be an element of the left hand side of (3.1.3). By $(C_{i+1,j})$ we obtain

$$\begin{aligned} a &\in [(y_1, \dots, y_{u-1}, x_i)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M] \colon y_u \\ &\subseteq (y_1, \dots, y_{u-1}, x_i)M \colon y_u + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M. \end{aligned}$$

On the other hand, (C_{ii}) gives

$$\begin{aligned} a &\in [(y_1, \dots, y_{u-1})M + \mathfrak{q}_i^2 M] \colon y_u \\ &\subseteq (y_1, \dots, y_{u-1})M \colon y_u + \mathfrak{q}_i M. \end{aligned}$$

Thus

$$\begin{aligned} a &\in [(y_1, \dots, y_{u-1}, x_i)M \colon y_u + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \cap [(y_1, \dots, y_{u-1})M \colon y_u + \mathfrak{q}_i M] \\ &= (y_1, \dots, y_{u-1})M \colon y_u + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M + (y_1, \dots, y_{u-1}, x_i)M \colon y_u \cap \mathfrak{q}_i M \\ &= (y_1, \dots, y_{u-1})M \colon y_u + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M + x_i M. \end{aligned}$$

Here we used (3.1.7) to show the last equation. Taking intersection with

$$[(y_1, \dots, y_{u-1})M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_u,$$

we have

$$\begin{aligned} a \in (y_1, \dots, y_{u-1})M \colon y_u + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M \\ + [(y_1, \dots, y_{u-1})M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_u \cap x_i M \\ = (y_1, \dots, y_{u-1})M \colon y_u + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M \\ + x_i\{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_u x_i\}. \end{aligned}$$

By applying $(E_{i+1,j})$ to a subsystem of parameters $y_1, \dots, y_u, x_i$ for $M/\mathfrak{q}_{i+1}M$, we have

$$[(y_1, \dots, y_{u-1})M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_u x_i = [(y_1, \dots, y_{u-1})M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon x_i.$$

Therefore $a \in (y_1, \dots, y_{u-1})M \colon y_u + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M$. The proof is completed. □

Step 7. *If $j > i$, then $(C_{i+1,j})$ implies (D_{ij}).*

Proof. We first show that

$$\text{(3.1.8)} \quad \begin{aligned} [(y_1, \dots, y_{u-1})M + x_i \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_u \\ = x_i\{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_u\} \\ + (y_1, \dots, y_{u-1})M \colon y_u. \end{aligned}$$

Let a be an element of the left hand side of (3.1.8) and put $y_u a = x_i b + c$ with $b \in \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M$ and $c \in (y_1, \dots, y_{u-1})M$. Then

$$\begin{aligned} b \in (y_1, \dots, y_u)M \colon x_i \cap \mathfrak{q}_i M \\ \subseteq (y_1, \dots, y_u)M \end{aligned}$$

because $x_i, \dots, x_d$ is a d-sequence on $M/(y_1, \dots, y_u)M$. We put $b = y_u a' + c'$ with $c' \in (y_1, \dots, y_{u-1})M$. Then $a' \in [(y_1, \dots, y_{u-1})M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_u$ and $a - x_i a' \in (y_1, \dots, y_{u-1})M \colon y_u$. Hence a is in the right hand side of (3.1.8) and the opposite inclusion is obvious.

Next we show (3.1.4). Let a be an element of M such that

$$x_i a \in [(y_1, \dots, y_{u-1})M + \mathfrak{q}_i \cdots \mathfrak{q}_j M] \colon y_u$$

and put $y_u x_i a = x_i b + b'$ with $b \in \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M$ and

$$b' \in (y_1, \dots, y_{u-1})M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M.$$

By applying $(C_{i+1,j})$ to a subsystem of parameters $y_1, \dots, y_{u-1}, x_i$ for $M/\mathfrak{q}_{i+1}M$, we have

$$\begin{aligned} y_u a - b \in [(y_1, \dots, y_{u-1})M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M] \colon x_i \\ \subseteq (y_1, \dots, y_{u-1})M \colon x_i + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M \end{aligned}$$

and hence

$$y_u a \in (y_1, \dots, y_{u-1})M \colon x_i + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M.$$

Therefore

$$\begin{aligned} x_i a \in [(y_1, \dots, y_{u-1})M + x_i \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_u \cap x_i M \\ = x_i\{[(y_1, \dots, y_{u-1})M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_u\} \\ + (y_1, \dots, y_{u-1})M \colon y_u \cap x_i M. \end{aligned}$$

Here we used (3.1.8). By applying (2.9.2) to a subsystem of parameters $y_1, \dots,$ y_u, x_i for $M/\mathfrak{q}_{i+1}M$, we have

$$(y_1, \dots, y_{u-1})M\colon y_u x_i = (y_1, \dots, y_{u-1})M\colon x_i.$$

Hence we have

$$\begin{aligned}(y_1, \dots, y_{u-1})M\colon y_u \cap x_i M &= x_i[(y_1, \dots, y_{u-1})M\colon y_u x_i] \\ &= x_i[(y_1, \dots, y_{u-1})M\colon x_i] \\ &\subseteq (y_1, \dots, y_{u-1})M.\end{aligned}$$

Hence $x_i a$ is in the right hand side of (3.1.4). Thus the proof is completed. ☐

Step 8. *$(E_{i,i+1})$ comes from $(B_{i,i+1})$, (E_{ii}) and $(E_{i+1,i+1})$.*

Proof. In the same way as Step 3, we may assume that $n_i = n_{i+1} = 1$. Let a be an element of the left hand side of (3.1.5). Then we have, by (E_{ii}),

$$\begin{aligned}a &\in [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i^2 M]\colon y_v y_u \\ &= [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i^2 M]\colon y_u.\end{aligned}$$

Therefore we put $y_u a = x_i^2 b + c$ with $c \in (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i\mathfrak{q}_{i+1}M$. On the other hand, $(E_{i+1,i+1})$ says that

$$\begin{aligned}a &\in [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} M]\colon y_v y_u \\ &\subseteq [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} M]\colon y_u.\end{aligned}$$

Thus

$$\begin{aligned}b &\in [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} M]\colon x_i^2 \\ &= [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} M]\colon x_i.\end{aligned}$$

Indeed, $x_i \in \mathfrak{a}(M)$ or $x_i \in \mathfrak{a}(M/\mathfrak{q}_{i+1}M)$. Therefore

$$y_u a = x_i^2 b + c \in (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i\mathfrak{q}_{i+1}M.$$

The proof is completed. ☐

Step 9. *If $j > i+1$, then (E_{ij}) is followed from (A_{ij}), (B_{ij}), $(C_{i+1,j})$, (D_{ij}) and $(E_{i+1,j})$.*

Proof. In the same way as Step 3, we may assume that $n_i = \cdots = n_j = 1$. Let a be an element of the left hand side of (3.1.5). By applying $(E_{i+1,j})$, we have

$$\begin{aligned}a &\in [(y_1, \dots, y_{v-1}, x_i, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M]\colon y_v y_u \\ &= [(y_1, \dots, y_{v-1}, x_i, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M]\colon y_u.\end{aligned}$$

Therefore

$$\begin{aligned}y_u a &\in [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i \cdots \mathfrak{q}_j M]\colon y_v \\ &\qquad \cap [(y_1, \dots, y_{v-1}, x_i, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M] \\ &= (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M \\ &\qquad + [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i \cdots \mathfrak{q}_j M]\colon y_v \cap x_i M \\ &= (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M \\ &\qquad + x_i\{[(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M]\colon y_v\}.\end{aligned}$$

Here we used (D_{ij}) to show the second equation. We put $y_u a = x_i b + c$ with

$$b \in [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_v$$

and

$$c \in (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M.$$

By applying $(C_{i+1,j})$ to a subsystem of parameters $y_1, \dots, y_{v-1}, y_u, \{x_\lambda \mid \lambda \in \Lambda\}, x_i$ for $M/\mathfrak{q}_{i+1}M$, we obtain

$$\begin{aligned} b &\in [(y_1, \dots, y_{v-1}, y_u, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M] \colon x_i \\ &\subseteq (y_1, \dots, y_{v-1}, y_u, x_\lambda \mid \lambda \in \Lambda)M \colon x_i + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M. \end{aligned}$$

Therefore

$$\begin{aligned} b \in\ & [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_v \\ & \cap [(y_1, \dots, y_{v-1}, y_u, x_\lambda \mid \lambda \in \Lambda)M \colon x_i + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \\ =\ & [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_v \\ & \cap (y_1, \dots, y_{v-1}, y_u, x_\lambda \mid \lambda \in \Lambda)M \colon x_i \\ & + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M. \end{aligned}$$

Since $j \geq i+2$, we have

$$\begin{aligned} &[(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_v \\ &\qquad \subseteq (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v + \mathfrak{q}_{i+2} M \end{aligned}$$

by using $(C_{i+1,j})$. Furthermore, we obtain

$$\begin{aligned} (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v &\subseteq (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v x_i \\ &= (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon x_i \\ &\subseteq (y_1, \dots, y_{v-1}, y_u, x_\lambda \mid \lambda \in \Lambda)M \colon x_i \end{aligned} \tag{3.1.9}$$

by using (2.9.2). Therefore

$$\begin{aligned} (y_1, \dots, y_{v-1}, y_u, x_\lambda \mid \lambda \in \Lambda)M \colon x_i \cap [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v + \mathfrak{q}_{i+2} M] \\ = (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v \\ + (y_1, \dots, y_{v-1}, y_u, x_\lambda \mid \lambda \in \Lambda)M \colon x_i \cap \mathfrak{q}_{i+2} M \\ \subseteq (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v + y_u M. \end{aligned}$$

Here we applied Lemma 2.2 to a d-sequence $x_i, \dots, x_d$ on

$$M/(y_1, \dots, y_{v-1}, y_u, x_\lambda \mid \lambda \in \Lambda)M.$$

Thus

$$\begin{aligned}
b &\in [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_v \\
&\qquad \cap [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v + \mathfrak{q}_{i+2} M] \\
&\qquad \cap (y_1, \dots, y_{v-1}, y_u, x_\lambda \mid \lambda \in \Lambda)M \colon x_i \\
&\quad + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M \\
&\subseteq [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_v \\
&\qquad \cap [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v + y_u M] \\
&\quad + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M \\
&= [(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_v \cap y_u M \\
&\quad + (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v \\
&\quad + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M \\
&= y_u \{[(y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_v y_u\} \\
&\quad + (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v \\
&\quad + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M \\
&= (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_v + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M.
\end{aligned}$$

Here we used $(E_{i+1,j})$ to show the last equation. Thus we have

$$y_u a = x_i b + c \in (y_1, \dots, y_{v-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i \cdots \mathfrak{q}_j M,$$

where we used (3.1.9) again. The opposite inclusion is obvious. □

Thus we finish the proof of Theorem 3.1.

The following corollary immediately comes from Theorem 3.1. We, however, can easily prove it if $j = i + 1$. See [21, Lemma 5.4].

Corollary 3.2. *With the same notation as Theorem 3.1, we have*

$$[(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \colon x_{i-1}^{n_{i-1}} = [(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \colon \mathfrak{q}_{i-1}$$

for any positive integers $2 \le i \le j \le d$, n_{i-1}, ... , n_j and a subsystem of parameters y_1, ... , y_u for $M/\mathfrak{q}_{i-1}M$.

Proof. We may assume that $n_{i-1} = 1$ by using (E_{ij}). Then we have

$$[(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \colon x_{i-1} \subseteq (y_1, \dots, y_u)M \colon x_{i-1} + \mathfrak{q}_i^{n_i - 1} \cdots \mathfrak{q}_j^{n_j} M$$

by applying (C_{ij}) to a subsystem of parameters y_1, ... , y_u, x_{i-1} for $M/\mathfrak{q}_i M$. Since x_{i-1}, ... , x_d is a d-sequence on $M/(y_1, \dots, y_u)M$,

$$(y_1, \dots, y_u)M \colon x_{i-1} \subseteq (y_1, \dots, y_u)M \colon \mathfrak{q}_{i-1}.$$

Therefore

$$\mathfrak{q}_{i-1}\{[(y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \colon x_{i-1}\} \subseteq (y_1, \dots, y_u)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M.$$

The opposite inclusion is obvious. □

We also need the following corollary for our Macaulayfication.

Corollary 3.3. *With the same notation as Theorem 3.1, we let k be an integer such that $1 \le k \le d$ and y_1, ... , y_u a subsystem of parameters for $M/\mathfrak{q}_k M$. If*

$$[(y_1, \dots, y_{u-1})M + \mathfrak{q}_k M] \colon y_u = (y_1, \dots, y_{u-1})M + \mathfrak{q}_k M,$$

then

$$(y_1, \dots, y_{u-1})M \colon y_u = (y_1, \dots, y_{u-1})M \tag{3.3.1}$$

and

$$\begin{aligned}[][(y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M] \colon y_u \\ = (y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i^{n_i} \cdots \mathfrak{q}_j^{n_j} M\end{aligned} \tag{3.3.2}$$

for any positive integers $k \le i \le j$, $n_i, \dots, n_j$ and $\Lambda \subseteq \{k, \dots, i-1\}$.

Proof. First we show that

$$(y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_u = (y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M \tag{3.3.3}$$

for any $\Lambda \subseteq \{k, \dots, d\}$. We work by descending induction on the number of elements in Λ. If $\Lambda = \{k, \dots, d\}$, then there is nothing to prove. Assume that $\Lambda \neq \{k, \dots, d\}$ and let l be the largest element of $\{k, \dots, d\} \setminus \Lambda$. Let a be an element of the left hand side of (3.3.3). Then

$$\begin{aligned} a &\in (y_1, \dots, y_{u-1}, x_l, x_\lambda \mid \lambda \in \Lambda)M \colon y_u \\ &= (y_1, \dots, y_{u-1}, x_l, x_\lambda \mid \lambda \in \Lambda)M \end{aligned}$$

by the induction hypothesis. We put $a = x_l b + c$ with

$$c \in (y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M.$$

Since $x_l \in \mathfrak{a}(M/\mathfrak{q}_{l+1}M)$ or $x_l \in \mathfrak{a}(M)$,

$$\begin{aligned} b &\in (y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M \colon y_u x_l \\ &= (y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M \colon x_l. \end{aligned}$$

Therefore we have $a \in (y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M$. By letting $\Lambda = \emptyset$, we obtain (3.3.1).

Next we prove (3.3.2) by descending induction on i. We may assume that $n_i = \cdots = n_j = 1$ by using (B_{ij}) of Theorem 3.1 and (3.3.1). In the case of $i = j$, (3.3.2) is included in (3.3.3).

Assume that $i < j$ and let a be an element of the left hand side of (3.3.2). By the induction hypothesis, we have

$$\begin{aligned} a &\in [(y_1, \dots, y_{u-1}, x_i, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M] \colon y_u \\ &= (y_1, \dots, y_{u-1}, x_i, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M. \end{aligned}$$

Therefore

$$\begin{aligned} a &\in [(y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i \cdots \mathfrak{q}_j M] \colon y_u \\ &\quad \cap [(y_1, \dots, y_{u-1}, x_i, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M] \\ &= [(y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i \cdots \mathfrak{q}_j M] \colon y_u \cap x_i M \\ &\quad + (y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M \\ &= x_i \{[(y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1} \cdots \mathfrak{q}_j M] \colon y_u\} \\ &\quad + (y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_{i+1}^2 \cdots \mathfrak{q}_j M \\ &= (y_1, \dots, y_{u-1}, x_\lambda \mid \lambda \in \Lambda)M + \mathfrak{q}_i \cdots \mathfrak{q}_j M. \end{aligned}$$

Here we used (D_{ij}) of Theorem 3.1 and the induction hypothesis to show the second and the last equations, respectively. □

4. Macaulayfication of Affine schemes

In this section, we construct a Macaulayfication of the affine scheme $X = \operatorname{Spec} A$ if A possesses a dualizing complex and is equidimensional. In this case the non-Cohen-Macaulay locus V of X is closed and A has a p-standard system of parameters of type s where s is an arbitrary integer such that $\dim V \leq s < \dim X$.

Let $\mathfrak{b}$ be an ideal of A and M an A-module. We agree that $R(\mathfrak{b})$ denotes the Rees algebra $\bigoplus_{n\geq 0} \mathfrak{b}^n$ and $R_M(\mathfrak{b})$ denotes the $R(\mathfrak{b})$-module $\bigoplus_{n\geq 0} \mathfrak{b}^n M$.

Theorem 4.1. *Let M be a finitely generated A-module of dimension $d > 0$ and $x_1, \dots, x_d$ a p-standard system of parameters of type s for M. Let*

$$\mathfrak{q}_i = (x_i, \dots, x_d);$$
$$\mathfrak{b}_{s,i} = \mathfrak{q}_i \cdots \mathfrak{q}_{s+1};$$
$$Y_{s,i} = \operatorname{Proj} R(\mathfrak{b}_{s,i})$$

and $\mathcal{F}_{s,i}$ be the coherent sheaf $[R_M(\mathfrak{b}_{s,i})]^\sim$ on $Y_{s,i}$ for all $1 \leq i \leq s+1$. Then

$$\text{(4.1.1)} \qquad \operatorname{depth}(\mathcal{F}_{s,t})_p \geq d - t + 1 \quad \text{for all closed points } p \text{ on } Y_{s,t}$$

for all $1 \leq t \leq s+1$. In particular, $\mathcal{F}_{s,1}$ is Cohen-Macaulay.

Furthermore, if $M/\mathfrak{q}_t M$ is Cohen-Macaulay for some $1 < t \leq s+1$, then $\mathcal{F}_{s,t}$ is already Cohen-Macaulay.

Corollary 4.2. *We assume that A is equidimensional and $d = \dim A > 0$. Let $x_1, \dots, x_d$ be a p-standard system of parameters of type s for A and put*

$$\mathfrak{q}_i = (x_i, \dots, x_d);$$
$$\mathfrak{b}_{s,i} = \mathfrak{q}_i \cdots \mathfrak{q}_{s+1};$$
$$Y_{s,i} = \operatorname{Proj} R(\mathfrak{b}_{s,i})$$

for all $1 \leq i \leq s+1$. Then the blowing-up $f_{s,1} \colon Y_{s,1} \to X = \operatorname{Spec} A$ is a Macaulayfication of X.

If $A/\mathfrak{q}_t$ is a Cohen-Macaulay ring for some $1 < t \leq s+1$, then the blowing-up $f_{s,t} \colon Y_{s,t} \to X$ is a Macaulayfication of X.

With notation above we obtain the sequence of blowing-ups, mentioned in Section 1,

$$Y_{s,1} \xrightarrow{g_{s,1}} Y_{s,2} \xrightarrow{g_{s,2}} \cdots \xrightarrow{g_{s,s-1}} Y_{s,s} \xrightarrow{g_{s,s}} Y_{s,s+1} \xrightarrow{f_{s,s+1}} X$$

where the center of $f_{s,s+1}$ is $\mathfrak{q}_{s+1}\mathcal{O}_X$ and the one of $g_{s,i}$ is $\mathfrak{q}_i \mathcal{O}_{Y_{s,i+1}}$ for all $i \leq s$. See [19, pp. 132–133]. Furthermore, $\mathfrak{q}_i \mathcal{O}_{Y_{s,i}}$ is invertible.

To prove Theorem 4.1, we show the following lemma by induction on t.

Lemma 4.3. *Let $x_t, \dots, x_d$ be a subsystem of parameters for M, where $1 \leq t < d = \dim M$. We fix an integer s such that $t - 1 \leq s < d$ and put*

$$\mathfrak{q}_i = (x_i, \dots, x_d);$$
$$\mathfrak{b}_i = \mathfrak{q}_i \cdots \mathfrak{q}_{s+1};$$
$$Y_i = \operatorname{Proj} R(\mathfrak{b}_i)$$

for all $t \leq i \leq s+1$. Let $\mathcal{F}_i$ be the coherent sheaf $[R_M(\mathfrak{b}_i)]^\sim$ on Y_i for all $t \leq i \leq s+1$. If the subsystem of parameters satisfies the following three conditions:

1. *the sequence $x_i, \dots, x_d$ is a d-sequence on $M/(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M$ for any positive integers $t \leq i < s+1$, $n_t, \dots, n_{i-1}$ and $\Lambda \subseteq \{t, \dots, i-1\}$;*

2. *the sequence* $x_{s+1}, \dots, x_d$ *is a* d^+*-sequence on* $M/(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M$ *for any positive integers* $n_t, \dots, n_s$ *and* $\Lambda \subseteq \{t, \dots, s\}$*;*
3. *the equality*

$$[(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M + \mathfrak{b}_i^{n_i} M] \colon x_{i-1}^{n_{i-1}} = [(x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M + \mathfrak{b}_i^{n_i} M] \colon \mathfrak{q}_{i-1}$$

holds for any positive integers $t+1 \leq i \leq s+1$, $n_t, \dots, n_i$ *and* $\Lambda \subseteq \{t, \dots, i-2\}$,

then

$$\text{depth}(\mathcal{F}_t)_p \geq d - t + 1 \quad \textit{for all closed points } p \textit{ on } Y_t. \tag{4.3.1}$$

It should be mentioned that the first and the third conditions say nothing if $t = s+1$.

Proof. First we note that $(\mathcal{F}_i)_p \neq 0$ and $\dim(\mathcal{F}_i)_p = \dim M$ for all closed points p on Y_i and $t \leq i \leq s+1$. Indeed, let $\mathfrak{p}$ be an associated prime ideal of M such that $\dim A/\mathfrak{p} = \dim M$. Then $\mathfrak{p}^* = \bigoplus_{n \geq 0} \mathfrak{p} \cap \mathfrak{b}_i^n$ is an associated prime ideal of $R_M(\mathfrak{b}_i)$: See [31, Proposition 1.1 (iii)]. Since $x_i, \dots, x_d$ is a subsystem of parameters for $A/\mathfrak{p}$, and hence is analytically independent on $A/\mathfrak{p}$, $\mathfrak{m}\mathfrak{b}_i^n$ contains $\mathfrak{p} \cap \mathfrak{b}_i^n$ for all $n \geq 0$. That is, all closed points on Y_i contain $\mathfrak{p}^*$. Furthermore $\dim R_M(\mathfrak{b}_i) = \dim R(\mathfrak{b}_i)/\mathfrak{p}^* = \dim A/\mathfrak{p} + 1$ [31, Corollary 1.6].

Assume that $t = s+1$ and let $R = R(\mathfrak{b}_{s+1})$. Then $\mathfrak{b}_{s+1} = \mathfrak{q}_{s+1}$ is generated by a d^+-sequence on M of length $d - s$. We find that, for all $0 \leq i \leq d-s$, the i^{th} local cohomology module of $R_M(\mathfrak{b}_{s+1})$ with respect to $\mathfrak{q}_{s+1}R + R_+$ is annihilated by some power of R_+ by [15, Proposition 4.9], whose brief proof is included in Appendix A. By passing through the completion of A and using [8, Satz 1], we obtain (4.3.1).

We note that we can obtain (4.3.1) without the theory of d^+-sequences if $s = d-1$. Indeed, it is easy to see $Y_d \cong \operatorname{Spec} A/H_{x_d}^0(A)$ and $\mathcal{F}_d = (M/H_{x_d}^0(M))^\sim$.

Next we assume that $t \leq s$. Then Y_t is the blowing-up of Y_{t+1} with respect to $\mathfrak{q}_t\mathcal{O}_{Y_{t+1}}$. Let q be a closed point on Y_t and p its image on Y_{t+1} under the blowing-up $Y_t \to Y_{t+1}$. Then p is also a closed point. Let $B = \mathcal{O}_{Y_{t+1},p}$, $N = (\mathcal{F}_{t+1})_p$ and $\mathfrak{n}$ be the maximal ideal of B. Since $\mathfrak{q}_{s+1}\mathcal{O}_{Y_{t+1}}$ is invertible, $\mathfrak{q}_{s+1}B$ is generated by x_i for some $t+1 \leq i \leq d$. In this case x_i is a regular element on B and on N. Thus we have the expression of $\mathcal{O}_{Y_t,q}$ and $(\mathcal{F}_t)_q$:

$$\mathcal{O}_{Y_t,q} = B[x_i/x_t]_{(\mathfrak{n}, f(x_i/x_t))}, \quad (\mathcal{F}_t)_q = \left[\bigcup_{n>0} \frac{(x_t, x_i)^n N}{x_t^n}\right]_{(\mathfrak{n}, f(x_i/x_t))} \tag{4.3.2}$$

or

$$\mathcal{O}_{Y_t,q} = B[x_t/x_i]_{(\mathfrak{n}, f(x_t/x_i))}, \quad (\mathcal{F}_t)_q = \left[\bigcup_{n>0} \frac{(x_t, x_i)^n N}{x_i^n}\right]_{(\mathfrak{n}, f(x_t/x_i))} \tag{4.3.3}$$

where f is a monic polynomial with coefficient in B.

We compute the local cohomology $H_{\mathfrak{q}_t}^p(N)$ of N with respect to $\mathfrak{q}_t$. Since $\mathfrak{q}_tB = (x_i, x_t)B$, $H_{\mathfrak{q}_t}^q(N) = 0$ if $q > 2$. Furthermore $H_{\mathfrak{q}_t}^0(N) = 0$ because x_i is regular on N. Let $\mathcal{F}_{t+1}^{(l)}$ be the coherent sheaf $[R_{M/x_t^lM}(\mathfrak{b}_{t+1})]^\sim$ on Y_{t+1} and $N^{(l)} = (\mathcal{F}_{t+1}^{(l)})_p$. Then the induction hypothesis says that

$$\text{depth}\, N, \text{depth}\, N^{(l)} \geq d - t.$$

There exists a short exact sequence

$$0 \to \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M : x_t^l}{\mathfrak{b}_{t+1}^n M + 0 :_M x_t^l} \xrightarrow{x_t^l} \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M}{x_t^l \mathfrak{b}_{t+1}^n M} \to \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M + x_t^l M}{x_t^l M} \to 0. \tag{4.3.4}$$

The left hand side of (4.3.4) is annihilated by $\mathfrak{q}_t$ because of (3). On the other hand, x_i is regular on the right hand side of (4.3.4). Indeed, if $i \le s$, then $x_i, \dots, x_d$ is a d-sequence on $M/x_t^l M$ and hence

$$x_t^l M : x_i \cap [x_t^l M + \mathfrak{b}_{t+1}^n M] \subseteq x_t^l M : x_i \cap (x_t^l, x_i, \dots, x_d)M = x_t^l M.$$

If $i \ge s+1$, then $x_{s+1}, \dots, x_d$ is a d^+-sequence on $M/x_t^l M$. Therefore

$$x_t^l M : x_i \cap [x_t^l M + \mathfrak{b}_{t+1}^n M] \subseteq x_t^l M : x_i \cap (x_t^l, x_{s+1}, \dots, x_d)M = x_t^l M.$$

By taking local cohomology with respect to x_i of (4.3.4), we obtain

$$H_{x_i}^1\left(\bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M}{x_t^l \mathfrak{b}_{t+1}^n M}\right) = H_{x_i}^1\left(\bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M + x_t^l M}{x_t^l M}\right) \tag{4.3.5}$$

and

$$H_{x_i}^0\left(\bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M}{x_t^l \mathfrak{b}_{t+1}^n M}\right) = \bigoplus_{n>0} \frac{\mathfrak{b}_{t+1}^n M : x_t^l}{\mathfrak{b}_{t+1}^n M + 0 :_M x_t^l}. \tag{4.3.6}$$

Taking localization of (4.3.5) at p, we have

$$H_{x_i}^1(N/x_t^l N) = H_{x_i}^1(N^{(l)}) = \operatorname{inj\,lim}_m N^{(l)}/x_i^m N^{(l)}.$$

The spectral sequence $E_2^{pq} = H_{x_i}^p H_{x_t}^q(-) \Rightarrow H_{(x_t,x_i)}^n(-)$ induces a short exact sequence

$$0 \to H_{x_i}^1 H_{x_t}^{p-1}(-) \to H_{(x_t,x_i)}^p(-) \to H_{x_i}^0 H_{x_t}^p(-) \to 0. \tag{4.3.7}$$

Hence

$$\begin{aligned} H_{\mathfrak{q}_t}^2(N) &= H_{x_i}^1 H_{x_t}^1(N) \\ &= H_{x_i}^1(\operatorname{inj\,lim}_l N/x_t^l N) \\ &= \operatorname{inj\,lim}_l H_{x_i}^1(N/x_t^l N) \\ &= \operatorname{inj\,lim}_l H_{x_i}^1(N^{(l)}) \\ &= \operatorname{inj\,lim}_{m,l} N^{(l)}/x_i^m N^{(l)}. \end{aligned}$$

Since x_i is regular on $N^{(l)}$ and $\operatorname{depth} N^{(l)} \ge d-t$,

$$H_{\mathfrak{n}}^p H_{\mathfrak{q}_t}^2(N) = \operatorname{inj\,lim}_{m,l} H_{\mathfrak{n}}^p(N^{(l)}/x_i^m N^{(l)}) = 0$$

for $p < d-t-1$.

Next we compute $H_{\mathfrak{q}_t}^1(N)$. Since $x_t, \dots, x_d$ is a d-sequence,

$$0 :_M x_t \cap \mathfrak{b}_{t+1}^n M \subseteq 0 :_M x_t \cap \mathfrak{q}_t M = 0$$

if $n > 0$. Therefore x_t is regular on $\bigoplus_{n>0} \mathfrak{b}_{t+1}^n M$ and hence on N, that is $H_{x_t}^0(N) = 0$. From (4.3.7), we have

$$H_{\mathfrak{q}_t}^1(N) = H_{x_i}^0 H_{x_t}^1(N) = \operatorname{inj\,lim}_l H_{x_i}^0(N/x_t^l N).$$

Taking localization and direct limit of (4.3.6), we find that $\mathfrak{q}_t H^1_{\mathfrak{q}_t}(N) = 0$.

We consider the spectral sequence $E_2^{pq} = H^p_{\mathfrak{n}} H^q_{\mathfrak{q}_t}(N) \Rightarrow H^n_{\mathfrak{n}}(N)$. We already know that $E_2^{pq} = 0$ if $q > 2$ or $q = 0$ and that $E_2^{p2} = 0$ if $p < d - t - 1$. Since $\operatorname{depth} N \geq d - t$, $E_2^{p1} = H^{p+1}_{\mathfrak{n}}(N) = 0$ for $p < d - t - 1$. Thus we obtain that

$$H^p_{\mathfrak{n}} H^q_{\mathfrak{q}_t}(N) = 0 \quad \text{if } q \neq 1,\, 2 \text{ or } p < d - t - 1$$

and

$$\mathfrak{q}_t H^1_{\mathfrak{q}_t}(N) = 0.$$

By using this, we compute the depth of $(\mathcal{F}_t)_q$. We assume that it has the expression (4.3.2). Let $L = N[T]/(x_t T - x_i)N[T]$, where T is an indeterminate and $N[T] = B[T] \otimes N$. Then

$$\bigcup_{n>0} \frac{(x_t, x_i)^n N}{x_t^n} \cong L/H^0_{x_t}(L).$$

Taking local cohomology with respect to $\mathfrak{q}_t$ of the short exact sequence

$$0 \to N[T] \xrightarrow{x_t T - x_i} N[T] \to L \to 0,$$

we obtain an exact sequence

$$0 \to H^1_{\mathfrak{q}_t}(N[T]) \to H^1_{\mathfrak{q}_t}(L) \to H^2_{\mathfrak{q}_t}(N[T]) \to H^2_{\mathfrak{q}_t}(N[T]) \to 0.$$

Recall that $H^1_{\mathfrak{q}_t}(N[T]) = H^1_{\mathfrak{q}_t}(N) \otimes B[T]$ is annihilated by $\mathfrak{q}_t$ and hence by $x_t T - x_i$. By using an exact sequence

$$0 \to H^1_{f(T)} H^{p-1}_{\mathfrak{n}}(-) \to H^p_{(\mathfrak{n}, f(T))}(-) \to H^0_{f(T)} H^p_{\mathfrak{n}}(-) \to 0,$$

we get $H^p_{(\mathfrak{n}, f(T))} H^q_{\mathfrak{q}_t}(N[T]) = 0$ if $q \neq 1$, 2 or $p < d - t$. Indeed, the monic polynomial $f(T)$ is regular on $H^p_{\mathfrak{n}} H^q_{\mathfrak{q}_t}(N[T]) = (H^p_{\mathfrak{n}} H^q_{\mathfrak{q}_t}(N)) \otimes B[T]$. Hence we obtain that

$$H^p_{(\mathfrak{n}, f(T))} H^1_{\mathfrak{q}_t}(L) = 0 \quad \text{for } p < d - t.$$

Taking local cohomology with respect to $\mathfrak{q}_t B[T] = (x_t, x_t T - x_i)B[T]$ of a short exact sequence,

$$0 \to H^0_{x_t}(L) \to L \to L/H^0_{x_t}(L) \to 0,$$

we have

$$H^1_{\mathfrak{q}_t}(L/H^0_{x_t}(L)) \cong H^1_{\mathfrak{q}_t}(L),$$

that is,

$$H^p_{(\mathfrak{n}, f(T))} H^1_{\mathfrak{q}_t}(L/H^0_{x_t}(L)) = 0 \quad \text{for } p < d - t.$$

Of course, $H^q_{\mathfrak{q}_t}(L/H^0_{x_t}(L)) = 0$ if $q \neq 1$. The spectral sequence

$$E_2^{pq} = H^p_{(\mathfrak{n}, f(T))} H^q_{\mathfrak{q}_t}(L/H^0_{x_t}(L)) \Rightarrow H^n_{(\mathfrak{n}, f(T))}(L/H^0_{x_t}(L))$$

says that $\operatorname{depth}(\mathcal{F}_t)_q \geq d - t + 1$. In the case of (4.3.3), we can also show that $\operatorname{depth}(\mathcal{F}_t)_q \geq d - t + 1$. The proof is completed. □

Now we return to Theorem 4.1.

Proof of Theorem 4.1. Because of Proposition 2.8, Corollaries 2.10 and 3.2, the subsystem of parameters $x_t, \dots, x_d$ satisfies the assumption of Lemma 4.3 for any $1 \le t \le s+1$. Hence we obtain (4.1.1).

Next we assume that $t \ge 2$ and that $M/\mathfrak{q}_t M$ is Cohen-Macaulay, that is, $x_1, \dots, x_{t-1}$ is a regular sequence on $M/\mathfrak{q}_t M$. Corollary 3.3 says that $x_1, \dots, x_{t-1}$ is a regular sequence on $M/\mathfrak{b}_{s,t}^n M$ for any $n > 0$ and M itself. Taking Koszul cohomology of the short exact sequence

$$0 \to R_M(\mathfrak{b}_{s,t}) \to \bigoplus_{n \ge 0} M_n \to \bigoplus_{n > 0} M/\mathfrak{b}_{s,t}^n M \to 0$$

where M_n denotes a copy of M, with respect to $x_1, \dots, x_{t-1}$, we obtain that

$$H^{t-1}(x_1, \dots, x_{t-1}; R_M(\mathfrak{b}_{s,t})) = R_{M/(x_1,\dots,x_{t-1})M}(\mathfrak{b}_{s,t})$$

and

$$H^i(x_1, \dots, x_{t-1}; R_M(\mathfrak{b}_{s,t})) = 0 \quad \text{for all } i < t-1.$$

Let $\mathcal{F}'_{s,t} = [R_{M/(x_1,\dots,x_{t-1})M}(\mathfrak{b}_{s,t})]^\sim$ and p be a closed point on $Y_{s,t}$. Two equations above mean that $x_1, \dots, x_{t-1}$ is a regular sequence on $(\mathcal{F}_{s,t})_p$ and

$$(\mathcal{F}'_{s,t})_p = (\mathcal{F}_{s,t})_p/(x_1, \dots, x_{t-1})(\mathcal{F}_{s,t})_p.$$

Lemma 4.3 also assures us that $\operatorname{depth}(\mathcal{F}'_{s,t})_p \ge d-t+1$. Therefore $\mathcal{F}_{s,t}$ is Cohen-Macaulay. The proof of Theorem 4.1 is completed. □

Proof of Corollary 4.2. We already know the depth of $Y_{s,t}$ at each closed point. Since $\dim A/\mathfrak{b}_{s,t} = s < \dim A$, there exists no minimal prime ideal of A containing $\mathfrak{b}_{s,t}$. Therefore $f_{s,t}$ is birational. Of course, the projective morphism $f_{s,t}$ is proper. Thus the proof is completed. □

Next we consider an $R(\mathfrak{b}_{s,i+1}^2)$-module $R_{\mathfrak{b}_{s,i+1}[\mathfrak{b}_{s,i+1}M:\, x_i]}(\mathfrak{b}_{s,i+1}^2)$ when $s > 0$. It is well-known that $Y_{s,i+1} \cong \operatorname{Proj} R(\mathfrak{b}_{s,i+1}^2)$. We identify them.

Lemma 4.4. *With the same notation as Theorem 4.1, let $\mathcal{G}_{s,i}$ be the coherent sheaf $[R_{\mathfrak{b}_{s,i+1}[\mathfrak{b}_{s,i+1}M:\, x_i]}(\mathfrak{b}_{s,i+1}^2)]^\sim$ on $Y_{s,i+1}$ for all $1 \le i \le s$. Then*

(4.4.1) $\quad \operatorname{depth}(\mathcal{G}_{s,t})_p \ge d-t+1$ *for all closed points p on $Y_{s,t+1}$*

for all $1 \le t \le s$. In particular $\mathcal{G}_{s,1}$ is Cohen-Macaulay.

Proof. We first show that x_t, x_{t+1} is a regular sequence on $\mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}M:\, x_t]$ for all $n > 0$. Indeed,

$$0:_M x_t \cap \mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}M:\, x_t] \subseteq 0:_M x_t \cap \mathfrak{q}_t M = 0.$$

Let a be an element of $\mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}M:\, x_t]$ such that

$$x_{t+1}a \in x_t \mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}M:\, x_t].$$

Then $x_t a \in x_t^2 M:\, x_{t+1} \cap \mathfrak{b}_{s,t+1}^{2n} M \subset x_t^2 M$ because $x_{t+1}, \dots, x_d$ is a d-sequence on $M/x_t^2 M$. Let $x_t a = x_t^2 b$. Then

$$b \in \mathfrak{b}_{s,t+1}^{2n} M:\, x_t^2 = \mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}M:\, x_t] + 0:_M x_t$$

because of Theorem 3.1. Hence

$$\begin{aligned} a &\in \{x_t\mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}M\colon x_t] + 0\colon{}_M x_t\} \cap \mathfrak{b}_{s,t+1}^{2n-1}M \\ &= x_t\mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}M\colon x_t] + 0\colon{}_M x_t \cap \mathfrak{b}_{s,t+1}^{2n-1}M \\ &= x_t\mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}M\colon x_t]. \end{aligned}$$

Next we consider the short exact sequence of $R\left(\mathfrak{b}_{s,t+1}^2\right)$-modules

(4.4.2)

$$0 \to \bigoplus_{n>0} \mathfrak{b}_{s,t+1}^{2n}M \to \bigoplus_{n>0} \mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}M\colon x_t] \to \bigoplus_{n>0} \frac{\mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}M\colon x_t]}{\mathfrak{b}_{s,t+1}^{2n}M} \to 0.$$

Since the right hand side of (4.4.2) is annihilated by $\mathfrak{q}_t$ and x_t, x_{t+1} is a regular sequence on the middle term of (4.4.2), we obtain that

$$H_{\mathfrak{q}_t}^q\left(\bigoplus_{n>0} \mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}M\colon x_t]\right) = \begin{cases} H_{\mathfrak{q}_t}^q(\bigoplus_{n>0} \mathfrak{b}_{s,t+1}^{2n}M) & \text{if } q \geq 2; \\ 0 & \text{if } q = 0 \text{ or } 1. \end{cases}$$

Let p be a closed point on $Y_{s,t+1}$. Then by taking localization of the equation above at p, we have

$$H_{\mathfrak{q}_t}^q((\mathcal{G}_{s,t})_p) = \begin{cases} H_{\mathfrak{q}_t}^q((\mathcal{F}_{s,t+1})_p) & \text{if } q \geq 2; \\ 0 & \text{if } q = 0 \text{ or } 1. \end{cases}$$

We know that $H_{\mathfrak{q}_t}^q((\mathcal{F}_{s,t+1})_p) = 0$ for $q > 2$ and $H_{\mathfrak{n}}^p H_{\mathfrak{q}_t}^2((\mathcal{F}_{s,t+1})_p) = 0$ for all $p < d-t-1$. Here $\mathfrak{n}$ denotes the maximal ideal of $\mathcal{O}_{Y_{s,t+1},p}$. The spectral sequence $E_2^{pq} = H_{\mathfrak{n}}^p H_{\mathfrak{q}_t}^q(-) \Rightarrow H_{\mathfrak{n}}^n(-)$ says (4.4.1). □

Corollary 4.5. *With the same notation as Corollary 4.2, we put*

$$Z_{s,i} = \operatorname{Proj} R\left(\mathfrak{b}_{s,i+1}[\mathfrak{b}_{s,i+1}\colon x_i]\right)$$

for all $1 \leq i \leq s$. *Then*

(4.5.1) $\quad \operatorname{depth} \mathcal{O}_{Z_{s,t},p} \geq d-t+1$ *for all closed points* p *on* $Z_{s,t}$

for all $1 \leq t \leq s$. *In particular the blowing-up* $Z_{s,1} \to X$ *is a Macaulayfication of* X.

The schemes $Z_{s,s}$ and $Z_{s,s-1}$ were essentially given by Brodmann [4, Satz 9.2] and the author [21, Theorem 5.6], respectively.

Proof. Since

$$\mathfrak{b}_{s,t+1}^n[\mathfrak{b}_{s,t+1}\colon x_t]^n \subseteq \mathfrak{b}_{s,t+1}^n[\mathfrak{b}_{s,t+1}^n\colon x_t^n] \subseteq \mathfrak{b}_{s,t+1}^{2n-1}[\mathfrak{b}_{s,t+1}\colon x_t]$$

for all $n > 0$, $\mathfrak{b}_{s,t+1}^2$ is a reduction of $\mathfrak{b}_{s,t+1}[\mathfrak{b}_{s,t+1}\colon x_t]$. Hence we obtain a finite morphism $h_{s,t}\colon Z_{s,t} \to Y_{s,t+1}$. Let p be a closed point on $Y_{s,t+1}$. Then Lemma 4.4 assures us that $\operatorname{depth}((h_{s,t})_*\mathcal{O}_{Z_{s,t}})_p \geq d-t+1$. Hence we obtain (4.5.1). We can show that the blowing-up $Z_{s,t} \to X$ is birational in the same way as Corollary 4.2. □

These Macaulayfications are different from each other. We find it by comparing the height and the analytic spread [23] of centers.

Lemma 4.6. *With the same notation as Corollary 4.2, the analytic spread of* $\mathfrak{b}_{s,i}$ *is equal to* $d-i+1$ *for all* $1 \leq i \leq s+1$. *Furthermore, the one of* $\mathfrak{b}_{s,i+1}[\mathfrak{b}_{s,i+1}\colon x_i]$ *is equal to* $d-i$ *for all* $1 \leq i \leq s$.

Proof. The ideal $\mathfrak{b}_{s,i}$ is generated by monomials of $x_i, \dots, x_d$ of degree $s-i+2$. Since $x_i, \dots, x_d$ is analytically independent,

$$\operatorname{length}(\mathfrak{q}_i^n/\mathfrak{m}\mathfrak{q}_i^n) \le \operatorname{length}(\mathfrak{b}_{s,i}^n/\mathfrak{m}\mathfrak{b}_{s,i}^n) \le \operatorname{length}(\mathfrak{q}_i^{n(s-i+2)}/\mathfrak{m}\mathfrak{q}_i^{n(s-i+2)})$$

for all $n > 0$. Thus the first assertion is proved.

Since $\mathfrak{b}_{s,i+1}^2$ is a reduction of $\mathfrak{b}_{s,i+1}[\mathfrak{b}_{s,i+1} : x_i]$, they have the same analytic spread. ☐

5. Macaulayfication of Noetherian schemes

We prove Theorem 1.1 in this section. First we consider a Macaulayfication of quasi-projective schemes.

Theorem 5.1. *Let A be a Noetherian ring possessing a dualizing complex and X a quasi-projective scheme over A. Then there exists a proper morphism $f\colon Y \to X$ with an open dense subset U of X such that*

1. *Y is Cohen-Macaulay;*
2. *$f^{-1}(U)$ is dense in Y;*
3. *$f|_{f^{-1}(U)}\colon f^{-1}(U) \to U$ is an isomorphism.*

That is, f is a Macaulayfication of X.

If finitely many Cohen-Macaulay points $p_1, \dots, p_n$ on X are given, then we may choose f and Y such that

4. *U contains $p_1, \dots, p_n$.*

Since a quasi-projective scheme is an open dense subscheme of a projective scheme, we may consider only projective schemes without loss of generality. Let $R = \bigoplus_{n\ge 0} R_n$ be a Noetherian graded ring such that R_0 has a dualizing complex and R is generated by R_1 as an R_0-algebra. We give a Macaulayfication of $X = \operatorname{Proj} R$. Of course, we assume that X is not Cohen-Macaulay and let V be the non-Cohen-Macaulay locus of X.

In this case R possesses a dualizing complex $D^\bullet$ with codimension function t as a graded ring.

The following lemma is an analogue of Lemma 2.4 and can be proved by the local duality theorem.

Lemma 5.2. *Let M be a finitely generated graded R-module and $\mathfrak{p}$ a homogeneous prime ideal of R.*

1. *If $\mathfrak{p} \supseteq \prod_{j>i} \operatorname{ann} H^j(\operatorname{Hom}(M, D^\bullet))$, then $t(\mathfrak{p}) > i$;*
2. *Assume that $t(\mathfrak{q}) = 0$ for all minimal prime ideals $\mathfrak{q}$ of R. Then $M_\mathfrak{p}$ is Cohen-Macaulay if and only if $\mathfrak{p} \not\supseteq \prod_{j>0} \operatorname{ann} H^j(\operatorname{Hom}(M, D^\bullet))$.*

Since $\operatorname{Proj} R/H^0_{R_+}(R) \cong X$, we may assume that $H^0_{R_+}(R) = 0$. Furthermore, we may assume that t is constant on the associated prime ideals of R. Indeed, let $(0) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ be the primary decomposition of the zero ideal of R. Of course, $\mathfrak{q}_i$ is homogeneous. For any integer i, let $\mathfrak{r}_i$ be the intersection of $\mathfrak{q}_j$ such that $\mathfrak{q}_j$ is an isolated component and $t(\sqrt{\mathfrak{q}_j}) = i$. Then

$$g\colon \coprod_i \operatorname{Proj} R/\mathfrak{r}_i \to X$$

is birational and proper. Indeed, $\mathfrak{r}_i = R$ for all but finitely many i, hence g is a finite morphism. On the other hand, g is an isomorphism on the Cohen-Macaulay locus $X \setminus V$ of X because a Cohen-Macaulay local ring is equidimensional and has no embedded prime ideal. If each $\operatorname{Proj} R/\mathfrak{r}_i$ has a Macaulayfication, then X also has a Macaulayfication. Therefore we assume that $t(\mathfrak{p}) = 0$ for all associated prime ideals $\mathfrak{p}$. In this case $t(\mathfrak{p}) = \operatorname{ht}\mathfrak{p}$ for any prime ideal.

Since X has a dualizing complex, $\dim X < \infty$. Let $d = \dim X$ and s be an integer such that $d - \operatorname{codim}(V, X) \leq s < d$. It should be mentioned that $d - \operatorname{codim}(V, X) \neq \dim V$ in general.

The following lemma is an easy consequence of prime avoidance. See the proof of Theorem 2.7.

Lemma 5.3. *With notation above, there exists a sequence $z_1, \dots, z_d$ of homogeneous elements in R satisfying the following statements*:

1. *if $\mathfrak{p}$ is a minimal prime ideal of $R/(z_i, \dots, z_d)R$ but not containing R_+, then* $\operatorname{ht}\mathfrak{p} = d - i + 1$;
2. $z_{s+1}, \dots, z_d \in \prod_{j>0} \operatorname{ann} H^j(D^\bullet)$;
3. $z_i \in \prod_{j>d-i} \operatorname{ann} H^j(\operatorname{Hom}(R/(z_{i+1}, \dots, z_d)R, D^\bullet)$ *for all $i \leq s$.*

Let $\mathfrak{b} = \prod_{i=1}^{s+1}(z_i, \dots, z_d)$ where $z_1, \dots, z_d$ is a sequence obtained above and $f\colon Y \to X$ be the blowing-up of X with respect to $\mathfrak{b}^\sim$. We show that f is a Macaulayfication of X. Since no minimal prime ideal of R contains z_d and hence $\mathfrak{b}$, f is birational and proper.

Let q be a closed point on Y and $\mathfrak{p} \subset R$ its image under f. If $\mathfrak{b} \not\subseteq \mathfrak{p}$, then $\prod_{j>0} \operatorname{ann} H^j(D^\bullet) \not\subseteq \mathfrak{p}$. Hence $\mathcal{O}_{Y,q} \cong R_{(\mathfrak{p})}$ is Cohen-Macaulay.

Assume that $\mathfrak{b} \subseteq \mathfrak{p}$. Then $z_t, \dots, z_d \in \mathfrak{p}$ and $z_{t-1} \notin \mathfrak{p}$ for some $1 \leq t \leq s+1$, where we set $z_0 = 1$. Choose an element $y \in R_1 \setminus \mathfrak{p}$ and let $x_i = z_i / y^{\deg z_i}$. Then

1. $\dim R_{(\mathfrak{p})}/(x_t, \dots, x_d)R_{(\mathfrak{p})} = \dim R_{(\mathfrak{p})} - (d - t + 1)$;
2. $x_{s+1}, \dots, x_d \in \mathfrak{a}(R_{(\mathfrak{p})})$;
3. $x_i \in \mathfrak{a}(R_{(\mathfrak{p})}/(x_{i+1}, \dots, x_d)R_{(\mathfrak{p})})$ for all $t \leq i \leq s$.
4. $\mathfrak{a}(R_{(\mathfrak{p})}/(x_t, \dots, x_d)R_{(\mathfrak{p})}) = R_{(\mathfrak{p})}$ if $t > 1$.

Hence $x_t, \dots, x_d$ is a subsystem of p-standard system of parameters for $R_{(\mathfrak{p})}$ and $R_{(\mathfrak{p})}/(x_t, \dots, x_d)R_{(\mathfrak{p})}$ is a Cohen-Macaulay ring if $t > 1$. Since

$$\mathfrak{b}R_{(\mathfrak{p})} = \prod_{i=t}^{s+1}(x_i, \dots, x_d)R_{(\mathfrak{p})},$$

Corollary 4.2 says that $\mathcal{O}_{Y,q}$ is Cohen-Macaulay.

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be finitely many Cohen-Macaulay points on X. Then we may choose z_d such that $z_d \notin \mathfrak{p}_1 \cup \dots \cup \mathfrak{p}_n$. Thus the proof of Theorem 5.1 is completed.

Proof of Theorem 1.1. Let A be a Noetherian ring possessing a dualizing complex and X a separated, of finite type scheme over $\operatorname{Spec} A$. By using Chow's Lemma [16, Théorème 5.6.1], we obtain a birational proper morphism $X' \to X$ such that X' is a quasi-projective scheme over $\operatorname{Spec} A$. Theorem 5.1 assures us that there exists a Macaulayfication $Y \to X'$ of X'. The composition $Y \to X' \to X$ is a Macaulayfication of X. □

6. Sharp's conjecture

We divide Theorem 1.2 into two statements.

Theorem 6.1. *Let A be a Noetherian ring possessing a dualizing complex with codimension function t. If*

1. $t(\mathfrak{p}) - \operatorname{ht}\mathfrak{p}$ *is locally constant on* $\operatorname{Spec} A$;
2. *A has no embedding prime ideal,*

then A is a homomorphic image of finite-dimensional Gorenstein ring.

Of course, an integral domain satisfies conditions (1) and (2).

Proof. Let $D^\bullet$ be a dualizing complex of A. If $\operatorname{Spec} A$ is disconnected, then A is a direct product of Noetherian rings which also satisfy (1) and (2). Therefore we may assume that $\operatorname{Spec} A$ is connected. In this case, condition (1) is equivalent to

1′ $A_\mathfrak{m}$ is equidimensional for all maximal ideals $\mathfrak{m}$ of A.

We assume that $t(\mathfrak{p}) = \operatorname{ht}\mathfrak{p}$ for all prime ideals $\mathfrak{p}$ of A instead of (1).

Let V be the non-Cohen-Macaulay locus of $X = \operatorname{Spec} A$. We work by induction on $s(A) = \dim A - \operatorname{codim}(V, X)$. Let $d = \dim A$ and $s = s(A)$. It should be mentioned that a Noetherian ring possessing a dualizing complex is of finite dimension.

Assume that $s > d$, that is, A is Cohen-Macaulay and let $K = H^0(D^\bullet)$. Then the idealization $A \ltimes K$ is a Gorenstein ring and A is its homomorphic image. Indeed, a prime ideal of $A \ltimes K$ has an expression $\mathfrak{p} \oplus K$ where $\mathfrak{p}$ is a prime ideal of A. Since $K_\mathfrak{p} = H^0(D^\bullet_\mathfrak{p})$ is the canonical module of $A_\mathfrak{p}$, $[A \ltimes K]_{\mathfrak{p}\oplus K} = A_\mathfrak{p} \ltimes K_\mathfrak{p}$ is a Gorenstein local ring [25].

Next we assume that $s \leq d$. In the same way as the preceding section, there exists a sequence $x_1, \dots, x_d$ of elements in A such that

1. $\operatorname{ht}(x_i, \dots, x_d) = d - i + 1$ for all i;
2. $x_{s+1}, \dots, x_d \in \prod_{j>0} \operatorname{ann} H^j(D^\bullet)$;
3. $x_i \in \prod_{j>d-i} \operatorname{ann} H^j(\operatorname{Hom}(A/(x_{i+1}, \dots, x_d), D^\bullet))$ for $1 \leq i \leq s$.

Let $\mathfrak{b} = \prod_{i=1}^{s+1}(x_i, \dots, x_d)$ and $R = R\left(\mathfrak{b}^{d-s-1}\right)$. Then a finitely generated A-algebra R has a dualizing complex.

We show that R satisfies (1′). Let $\mathfrak{M}$ be a maximal ideal of R. Since all minimal prime ideals are homogeneous, we may assume that $\mathfrak{M}$ is homogeneous. Let $\mathfrak{m} = \mathfrak{M} \cap A$. Then $\mathfrak{M} = \mathfrak{m}R + R_+$. We may assume that A is a local ring with maximal ideal $\mathfrak{m}$ by passing though the localization. If $\mathfrak{b} \not\subseteq \mathfrak{m}$, then $R_\mathfrak{M} \cong A[T]_{(\mathfrak{m},T)}$, where T is an indeterminate, and it is equidimensional. Assume that $\mathfrak{b} \subseteq \mathfrak{m}$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the associated prime ideals of A and $\mathfrak{p}_i^* = \bigoplus_{n\geq 0} \mathfrak{b}^{n(d-s-1)} \cap \mathfrak{p}_i$. Then $\mathfrak{p}_1^*, \dots, \mathfrak{p}_m^*$ are the associated prime ideals of R and $\dim R/\mathfrak{p}_i^* = \dim A/\mathfrak{p}_i + 1 = d + 1$ for all i. See Proposition 1.1 and Corollary 1.6 of [31]. In particular, we find $\dim R = d + 1$. Proposition 1.1 of [31] also assures us that R satisfies (2).

Next we show that R is Cohen-Macaulay or $s(R) < s$. Assume that R is not Cohen-Macaulay and let $\mathfrak{P}$ be a prime ideal of R such that $R_\mathfrak{P}$ is not Cohen-Macaulay and $\operatorname{ht}\mathfrak{P} = (d+1) - s(R)$. Then $\mathfrak{P}$ must be homogeneous. By the same argument as the preceding section, we have $\mathfrak{P} \supseteq \mathfrak{b}^{d-s-1}R + R_+$. Let $\mathfrak{p} = \mathfrak{P} \cap A$. Then $\mathfrak{P} = \mathfrak{p}R + R_+$ and $\operatorname{ht}\mathfrak{P} = \operatorname{ht}\mathfrak{p} + 1$. Since $\mathfrak{p} \supseteq (x_{s+1}, \dots, x_d)$, the height of $\mathfrak{p}$ must be at least $d - s$. If $\operatorname{ht}\mathfrak{p} = d - s$, then $x_s \notin \mathfrak{p}$. Since $x_{s+1}, \dots, x_d$ is a d^+-sequence on $A_\mathfrak{p}$, $R_\mathfrak{p} = R\left((x_{s+1}, \dots, x_d)^{d-s-1}A_\mathfrak{p}\right)$ is Cohen-Macaulay. See

[15, Theorem 7.11] or Theorem A.5 in appendix A. It is a contradiction. Hence $s(R) = (d+1) - \operatorname{ht}\mathfrak{P} = d - \operatorname{ht}\mathfrak{p} < s$. In particular, if $s = 0$, then R must be Cohen-Macaulay.

The induction hypothesis says that R is a homomorphic image of a finite-dimensional Gorenstein ring and hence $A \cong R/R_+$ is also. Thus the proof is completed. □

Corollary 6.2. *A Noetherian local ring possessing a dualizing complex is a homomorphic image of a Gorenstein local ring.*

Proof. Let A be a Noetherian local ring possessing a dualizing complex. By [1, Lemma 3.1], we may assume that A satisfies Serre's (S_2)-condition. On the other hand, Ogoma [24, Lemma 4.1] showed that such a ring is equidimensional and has no embedded prime ideal. Therefore A satisfies the assumption of Theorem 6.1. □

7. Acknowledgment

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Appendix A. d^+-sequences

In this appendix, we select a few results from [15]. The author would like to thank Professor Shiro Goto and Professor Kikumichi Yamagishi for permission to refer to their results.

Let A be a commutative ring, M an A-module and $x_1, \dots, x_s$ a d^+-sequence on M. We put $\mathfrak{q} = (x_1, \dots, x_s)$.

Theorem A.1 ([15, Lemma 2.2])**.** *Assume that $s \geq 2$. For any integers $n_1, \dots, n_{s-1} \geq 2$, we have*

$$(x_1^{n_1}, \dots, x_{s-1}^{n_{s-1}})M \colon x_s = \sum_{\Lambda \subseteq \{1,\dots,s-1\}} \left(\prod_{\lambda \in \Lambda} x_\lambda^{n_\lambda - 1} \right) [(x_\lambda \mid \lambda \in \Lambda)M \colon x_s].$$

Here we set $(x_\lambda \mid \lambda \in \Lambda) = (0)$ and $\prod_{\lambda \in \Lambda} x_\lambda^{n_\lambda - 1} = 1$ if $\Lambda = \emptyset$.

Proof. We work by induction on s. Let $a \in x_1^{n_1}M \colon x_2$ and put $x_2 a = x_1^{n_1} b$. Then $b \in x_2 M \colon x_1^{n_1} = x_2 M \colon x_1$. If we put $x_1 b = x_2 a'$, then $a' \in x_1 M \colon x_2$ and $a - x_1^{n_1-1} a' \in 0 \colon_M x_2$. Hence $a \in x_1^{n_1-1}[x_1 M \colon x_2] + 0 \colon_M x_2$.

Assume that $s > 2$. Since x_{s-1}, x_s is a d^+-sequence on $M/(x_1^{n_1}, \dots, x_{s-2}^{n_{s-2}})M$,

$$\begin{aligned}(x_1^{n_1}, \dots, x_{s-1}^{n_{s-1}})M \colon x_s = x_{s-1}^{n_{s-1}-1}&[(x_1^{n_1}, \dots, x_{s-2}^{n_{s-2}}, x_{s-1})M \colon x_s] \\ &+ (x_1^{n_1}, \dots, x_{s-2}^{n_{s-2}})M \colon x_s.\end{aligned}$$

Since $x_1, \dots, x_{s-2}, x_s$ is a d^+-sequence on $M/x_{s-1}M$ and on M itself, we obtain

$$\begin{aligned}
&(x_1^{n_1}, \dots, x_{s-1}^{n_{s-1}})M \colon x_s \\
&= x_{s-1}^{n_{s-1}-1}\left\{\sum_{\Lambda\subseteq\{1,\dots,s-2\}}\left(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda-1}\right)[(x_{s-1}, x_\lambda \mid \lambda\in\Lambda)M \colon x_s]\right\} \\
&\quad + \sum_{\Lambda\subseteq\{1,\dots,s-2\}}\left(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda-1}\right)[(x_\lambda \mid \lambda\in\Lambda)M \colon x_s] \\
&= \sum_{\Lambda\subseteq\{1,\dots,s-1\}}\left(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda-1}\right)[(x_\lambda \mid \lambda\in\Lambda)M \colon x_s]
\end{aligned}$$

by the induction hypothesis. □

Theorem A.2 ([15, Theorem 2.6])**.** *For any integers $1 \le k \le s$, $n \ge 0$ and $n_1, \dots, n_k \ge 1$, we have*

$$(A_k) \qquad (x_1^{n_1}, \dots, x_k^{n_k})M \cap \mathfrak{q}^n M = \sum_{i=1}^{k} x_i^{n_i}\mathfrak{q}^{n-n_i}M.$$

Proof. We prove (A_k) and

$$(B_k) \qquad \begin{aligned}
&[(x_1^{n_1}, \dots, x_{k-1}^{n_{k-1}})M \colon x_k + x_k M] \cap \mathfrak{q}^n M \\
&= \sum_{i=1}^{k-1} x_i^{n_i}\mathfrak{q}^{n-n_i}M + x_k\mathfrak{q}^{n-1}M \\
&\quad + \sum_{\substack{\Lambda\subseteq\{1,\dots,k-1\} \\ \sum_{\lambda\in\Lambda}(n_\lambda-1)\ge n}}\left(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda-1}\right)[(x_\lambda \mid \lambda\in\Lambda)M \colon x_k]
\end{aligned}$$

for $1 \le k \le s$ by induction on k. Here we set $\sum_{\lambda\in\Lambda}(n_\lambda - 1) = 0$ if $\Lambda = \emptyset$.

First we show (A_1) by induction on n_1. Lemma 2.2 says that $x_1M \cap \mathfrak{q}^n M = x_1\mathfrak{q}^{n-1}M$ for all $n \ge 0$. Assume that $n > n_1 > 1$ and let $x_1^{n_1}a \in \mathfrak{q}^n M$. Then $x_1^{n_1}a \in x_1^{n_1-1}M \cap \mathfrak{q}^n M = x_1^{n_1-1}\mathfrak{q}^{n-n_1+1}M$ by the induction hypothesis. Let $x_1^{n_1}a = x_1^{n_1-1}a'$ with $a' \in \mathfrak{q}^{n-n_1+1}M$. Then $x_1a - a' \in 0 :_M x_1 \cap \mathfrak{q}M = 0$, that is, $a' = x_1a$. Therefore $a' \in x_1M \cap \mathfrak{q}^{n-n_1+1}M = x_1\mathfrak{q}^{n-n_1}M$ and hence $x_1^{n_1}a \in x_1^{n_1}\mathfrak{q}^{n-n_1}M$. If $n_1 \ge n$, then (A_1) is trivial.

Next we show (B_1). If $n = 0$, then it is trivial. Assume that $n > 0$. Let a be an element of the left hand side. Then $x_1a \in x_1^2M \cap \mathfrak{q}^{n+1}M = x_1^2\mathfrak{q}^{n-1}M$. If we put $x_1a = x_1^2b$ with $b \in \mathfrak{q}^{n-1}M$, then $a - x_1b \in 0 :_M x_1 \cap \mathfrak{q}^n M = 0$. Therefore $a \in x_1\mathfrak{q}^{n-1}M$.

Assume that $k \ge 2$. We show (B_k) by induction on n. If $n = 0$, then (B_k) coincides with Theorem A.1. Let $n > 0$. If $n_i = 1$, then (B_k) comes from (B_{k-1}) applied to a d^+-sequence $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_s$ on M/x_iM. For example, if

$n_1 = 1$, then

$$\begin{aligned}
&[(x_1, x_2^{n_2}, \dots, x_{k-1}^{n_{k-1}})M\colon x_k + x_k M] \cap [x_1 M + \mathfrak{q}^n M] \\
&\quad = x_1 M + \sum_{i=2}^{k-1} x_i^{n_i} \mathfrak{q}^{n-n_i} M + x_k \mathfrak{q}_k^{n_k - 1} M \\
&\qquad + \sum_{\substack{\Lambda \subseteq \{2,\dots,k-1\} \\ \sum_{\lambda\in\Lambda}(n_\lambda - 1) \geq n}} \left(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda - 1} \right) [(x_1, x_\lambda \mid \lambda \in \Lambda)M\colon x_k].
\end{aligned}$$

Taking intersection with $\mathfrak{q}^n M$, we obtain (B_1). Thus we may assume that $n_1, \dots, n_{k-1} \geq 2$. Let a be an element of the left hand side of (B_k). Since $a \in \mathfrak{q}^{n-1}M$,

$$\begin{aligned}
a &\in [(x_1^{n_1}, \dots, x_{k-1}^{n_{k-1}})M\colon x_k + x_k M] \cap \mathfrak{q}^{n-1} M \\
&= \sum_{i=1}^{k-1} x_i^{n_i} \mathfrak{q}^{n-n_i-1} M + x_k \mathfrak{q}^{n-2} M \\
&\quad + \sum_{\substack{\Lambda \subseteq \{1,\dots,k-1\} \\ \sum_{\lambda\in\Lambda}(n_\lambda - 1) \geq n-1}} \left(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda - 1} \right) [(x_\lambda \mid \lambda \in \Lambda)M\colon x_k].
\end{aligned}$$

We put

$$a = b + \sum_{\substack{\Lambda \subseteq \{1,\dots,k-1\} \\ \sum_{\lambda\in\Lambda}(n_\lambda - 1) \geq n-1}} \left(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda - 1} \right) c_\Lambda$$

with $b \in \sum_{i=1}^{k-1} x_i^{n_i} \mathfrak{q}^{n-n_i-1} M + x_k \mathfrak{q}^{n-2} M$ and $c_\Lambda \in (x_\lambda \mid \lambda \in \Lambda)M\colon x_k$. If $\sum_{\lambda\in\Lambda}(n_\lambda - 1) = n-1$, then $(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda - 1}) c_\Lambda \in (x_\lambda^{n_\lambda} \mid \lambda \in \Lambda)M$. Indeed, by renumbering $x_1, \dots, x_{k-1}$, we may assume that $\Lambda = \{1, \dots, p\}$. Then

$$a \in \mathfrak{q}^n M \subset (x_1^{n_1}, \dots, x_p^{n_p}, x_{p+1}, \dots, x_s)M.$$

Let $\Lambda' \subset \{1, \dots, k-1\}$ such that $\sum_{\lambda\in\Lambda'}(n_\lambda - 1) \geq n-1$. If $\Lambda' \subsetneq \Lambda$, then

$$\sum_{\lambda\in\Lambda'} (n_\lambda - 1) < \sum_{\lambda\in\Lambda} (n_\lambda - 1) = n - 1,$$

which is a contradiction. Therefore if $\Lambda' \neq \Lambda$, then $\Lambda' \cap \{p+1, \dots, k-1\} \neq \emptyset$ and hence $(\prod_{\lambda\in\Lambda'} x_\lambda^{n_\lambda - 1}) c_{\Lambda'} \in (x_{p+1}, \dots, x_{k-1})M$. On the other hand, it is easy to show that $(x_1^{n_1 - 1} \cdots x_p^{n_p - 1}) c_\Lambda \in (x_1^{n_1}, \dots, x_p^{n_p})M\colon x_k$. Therefore.

$$\begin{aligned}
(x_1^{n_1-1} \cdots x_p^{n_p-1}) c_\Lambda &= a - b - \sum_{\Lambda' \neq \Lambda} \left(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda - 1} \right) c_{\Lambda'} \\
&\in (x_1^{n_1}, \dots, x_p^{n_p})M\colon x_k \cap (x_1^{n_1}, \dots, x_p^{n_p}, x_{p+1}, \dots, x_s)M \\
&= (x_1^{n_1}, \dots, x_p^{n_p})M.
\end{aligned}$$

Here we applied Lemma 2.2 to a d-sequence $x_1^{n_1}, \ldots, x_p^{n_p}, x_k, x_{p+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_s$ on M. Thus

$$\begin{aligned} a - \sum_{\sum_{\lambda\in\Lambda}(n_\lambda-1)\geq n} \left(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda-1}\right) c_\Lambda &= b + \sum_{\sum_{\lambda\in\Lambda}(n_\lambda-1)=n-1} \left(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda-1}\right) c_\Lambda \\ &\in (x_1^{n_1}, \ldots, x_{k-1}^{n_{k-1}}, x_k)M \cap \mathfrak{q}^n M \\ &= (x_1^{n_1}, \ldots, x_{k-1}^{n_{k-1}}, x_k)M \\ &\quad \cap [\mathfrak{q}^n M + x_k M] \cap \mathfrak{q}^n M \\ &= \left[\sum_{i=1}^{k-1} x_i^{n_i} \mathfrak{q}^{n-n_i} M + x_k M\right] \cap \mathfrak{q}^n M \\ &= \sum_{i=1}^{k-1} x_i^{n_i} \mathfrak{q}^{n-n_i} M + x_k \mathfrak{q}^{n-1} M. \end{aligned}$$

Here we applied (A_{k-1}) to a d^+-sequence $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_s$ on M/x_kM. Thus (B_k) is proved.

Next we show (A_k) by induction on n_k. If $n_k = 1$, then we obtain (A_k) by applying (A_{k-1}) to a d^+-sequence $x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_s$ on M/x_kM. Assume that $n_k > 1$ and let $a \in (x_1^{n_1}, \ldots, x_k^{n_k})M \cap \mathfrak{q}^n M$. Then

$$\begin{aligned} a &\in (x_1^{n_1}, \ldots, x_{k-1}^{n_{k-1}}, x_k^{n_k-1})M \cap \mathfrak{q}^n M \\ &= \sum_{i=1}^{k-1} x_i^{n_i} \mathfrak{q}^{n-n_i} M + x_k^{n_k-1} \mathfrak{q}^{n-n_k+1} M. \end{aligned}$$

Let $a = b + x_k^{n_k-1} c$ with $b \in \sum_{i=1}^{k-1} x_i^{n_i} \mathfrak{q}^{n-n_i} M$ and $c \in \mathfrak{q}^{n-n_k+1} M$. Then

$$c \in (x_1^{n_1}, \ldots, x_k^{n_k})M : x_k^{n_k-1}.$$

If we put $x_k^{n_k-1} c = x_k^{n_k} c' + d$ with $d \in (x_1^{n_1}, \ldots, x_{k-1}^{n_{k-1}})M$, then

$$\begin{aligned} c - x_k c' &\in (x_1^{n_1}, \ldots, x_{k-1}^{n_{k-1}})M : x_k^{n_k-1} \\ &= (x_1^{n_1}, \ldots, x_{k-1}^{n_{k-1}})M : x_k. \end{aligned}$$

Therefore

$$c \in (x_1^{n_1}, \ldots, x_{k-1}^{n_{k-1}})M : x_k + x_k M.$$

By using (B_k), we obtain

$$\begin{aligned} c &\in [(x_1^{n_1}, \ldots, x_{k-1}^{n_{k-1}})M : x_k + x_k M] \cap \mathfrak{q}^{n-n_k+1} M \\ &= \sum_{i=1}^{k-1} x_i^{n_i} \mathfrak{q}^{n-n_k-n_i+1} M + x_k \mathfrak{q}^{n-n_k} M \\ &\quad + \sum_{\substack{\Lambda\subseteq\{1,\ldots,k-1\} \\ \sum_{\lambda\in\Lambda}(n_\lambda-1)\geq n-n_k+1}} \left(\prod_{\lambda\in\Lambda} x_\lambda^{n_\lambda-1}\right) [(x_\lambda \mid \lambda \in \Lambda)M : x_k] \end{aligned}$$

and hence $a = b + x_k^{n_k-1} c \in \sum_{i=1}^{k} x_i^{n_i} \mathfrak{q}^{n-n_i} M$. □

From now on, we assume that $x_1, \dots, x_s$ is a d^+-sequence on A itself. Theorems A.3–A.5 still hold if A is not Noetherian. We, however, assume that A is Noetherian for the sake of simplicity. We compute local cohomology modules of the Rees algebra $R(\mathfrak{q}) = A[\mathfrak{q}t]$, where t is an indeterminate. Let $G(\mathfrak{q}) = R(\mathfrak{q})/\mathfrak{q}R(\mathfrak{q})$.

Theorem A.3 ([15, Proposition 4.3]). *If $p < s$, then*

(A.3.1) $$[H^p_{\mathfrak{q}t}(G(\mathfrak{q}))]_n = 0 \quad \text{for } n \neq -p.$$

In particular $H^0_{\mathfrak{q}t}(G(\mathfrak{q})) = [H^0_{\mathfrak{q}t}(G(\mathfrak{q}))]_0 \cong 0 :_A x_1$.

Proof. We show that

$$[H^p_{(x_1t,\dots,x_qt)}(G(\mathfrak{q}))]_n = 0 \quad \text{if } p < q \text{ and } n \neq -p$$

for all $1 \le q \le s$ by induction on q. Let $\overline{at^n} \in H^0_{x_1t}(G(\mathfrak{q}))$, that is, $a \in \mathfrak{q}^n$ and $x_1^l a \in \mathfrak{q}^{n+l+1}$ for some $l > 0$. Then $x_1^l a \in (x_1^l) \cap \mathfrak{q}^{n+l+1} = x_1^l \mathfrak{q}^{n+1}$. Let $x_1^l a = x_1^l a'$ with $a' \in \mathfrak{q}^{n+1}$. If $n > 0$, then $a - a' \in 0 :_A x_1 \cap \mathfrak{q} = 0$, that is, $\overline{at^n} = 0$ in $G(\mathfrak{q})$. If $n = 0$, then $a \in 0 :_A x_1 + \mathfrak{q}$. On the other hand, if $a \in 0 :_A x_1 + \mathfrak{q}$, then $\overline{a} \in H^0_{x_1t}(G(\mathfrak{q}))$. Hence

$$H^0_{x_1t}(G(\mathfrak{q})) = (0 : x_1 + \mathfrak{q})/\mathfrak{q} \cong 0 : x_1.$$

Of course, $[H^0_{x_1t}(G(\mathfrak{q}))]_n = 0$ if $n < 0$.

Assume that $q > 1$. From the spectral sequence $E_2^{rs} = H^r_{x_qt} H^s_{(x_1t,\dots,x_{q-1}t)}(-) \Rightarrow H^n_{(x_1t,\dots,x_qt)}(-)$, we obtain an exact sequence

$$0 \to H^1_{x_qt} H^{p-1}_{(x_1t,\dots,x_{q-1}t)}(-) \to H^p_{(x_1t,\dots,x_qt)}(-) \to H^0_{x_qt} H^p_{(x_1t,\dots,x_{q-1}t)}(-) \to 0.$$

The induction hypothesis says that x_qt annihilates $H^p_{(x_1t,\dots,x_{q-1}t)}(G(\mathfrak{q}))$ for all $p < q-1$. Therefore, if $p < q-1$, then

$$H^p_{(x_1t,\dots,x_qt)}(G(\mathfrak{q})) = H^0_{x_qt} H^p_{(x_1t,\dots,x_{q-1}t)}(G(\mathfrak{q})) = H^p_{(x_1t,\dots,x_{q-1}t)}(G(\mathfrak{q}))$$

is concentrated in degree $-p$. Moreover

$$\begin{aligned} H^{q-1}_{(x_1t,\dots,x_qt)}(G(\mathfrak{q})) &= H^0_{x_qt} H^{q-1}_{(x_1t,\dots,x_{q-1}t)}(G(\mathfrak{q})) \\ &= \operatorname{inj\,lim}_m H^0_{x_qt}\left(\frac{G(\mathfrak{q})}{((x_1t)^m,\dots,(x_{q-1}t)^m)G(\mathfrak{q})}(m(q-1))\right) \\ &= \operatorname{inj\,lim}_m \bigcup_l \frac{((x_1t)^m,\dots,(x_{q-1}t)^m)G(\mathfrak{q}) : (x_qt)^l}{((x_1t)^m,\dots,(x_{q-1}t)^m)G(\mathfrak{q})}(m(q-1)). \end{aligned}$$

Let $\overline{at^{n+m(q-1)}} \in ((x_1t)^m,\dots,(x_{q-1}t)^m)G(\mathfrak{q}) : (x_qt)^l$, that is, $a \in \mathfrak{q}^{n+m(q-1)}$ and

$$x_q^l a \in (x_1^m,\dots,x_{q-1}^m)\mathfrak{q}^{n+m(q-2)+l} + \mathfrak{q}^{n+m(q-1)+l+1}.$$

If we put $a = b + c$ with $b \in (x_1^m,\dots,x_{q-1}^m)\mathfrak{q}^{n+m(q-2)+l}$ and $c \in \mathfrak{q}^{n+m(q-1)+l+1}$, then

$$\begin{aligned} c &\in (x_1^m,\dots,x_{q-1}^m, x_q^l) \cap \mathfrak{q}^{n+m(q-1)+l+1} \\ &= (x_1^m,\dots,x_{q-1}^m)\mathfrak{q}^{n+m(q-2)+l+1} + x_q^l \mathfrak{q}^{n+m(q-1)+1}. \end{aligned}$$

If we put $c = b' + x_q^l a'$ with $b' \in (x_1^m,\dots,x_{q-1}^m)\mathfrak{q}^{n+m(q-2)+l+1}$ and $a' \in \mathfrak{q}^{n+m(q-1)+1}$, then $\overline{at^{n+m(q-1)}} = \overline{(a-a')t^{n+m(q-1)}}$. Therefore, by replacing a by $a - a'$, we may assume that $x_q^l a \in (x_1^m,\dots,x_{q-1}^m)\mathfrak{q}^{n+m(q-2)+l}$. Then

$$a \in (x_1^m,\dots,x_{q-1}^m) : x_q \cap \mathfrak{q}^{n+m(q-1)}.$$

If $n > -q+1$, then $\mathfrak{q}^{n+m(q-1)} \subset (x_1^m, \ldots, x_{q-1}^m, x_q, \ldots, x_s)$ and hence

$$\begin{aligned} a &\in (x_1^m, \ldots, x_{q-1}^m) \colon x_q \cap (x_1^m, \ldots, x_{q-1}^m, x_q, \ldots, x_s) \cap \mathfrak{q}^{n+m(q-1)} \\ &= (x_1^m, \ldots, x_{q-1}^m) \cap \mathfrak{q}^{n+m(q-1)} \\ &= (x_1^m, \ldots, x_{q-1}^m)\mathfrak{q}^{n+m(q-2)}. \end{aligned}$$

That is, $\overline{at^{n+m(q-1)}} \in ((x_1t)^m, \ldots, (x_{q-1}t)^m)G(\mathfrak{q})$.

If $n < -q+1$, then $(m-1)(q-1) \geq n + m(q-1) + 1$. Hence we have

$$\begin{aligned} a &\in (x_1^m, \ldots, x_{q-1}^m) \colon x_q \\ &\subseteq \sum_{\Lambda \subsetneq \{1,\ldots,q-1\}} \left(\prod_{\lambda\in\Lambda} x_\lambda^{m-1} \right) [(x_\lambda \mid \lambda \in \Lambda) \colon x_q] + \mathfrak{q}^{n+m(q-1)+1} \end{aligned}$$

by using Theorem A.1. Therefore

$$\begin{aligned} x_1 \cdots x_{q-1} a &\in [(x_1^{m+1}, \ldots, x_{q-1}^{m+1}) + \mathfrak{q}^{n+(m+1)(q-1)+1}] \cap \mathfrak{q}^{n+(m+1)(q-1)} \\ &= (x_1^{m+1}, \ldots, x_{q-1}^{m+1})\mathfrak{q}^{n+(m+1)(q-2)} + \mathfrak{q}^{n+(m+1)(q-1)+1}. \end{aligned}$$

Indeed, if $k \in \{1, \ldots, q-1\} \setminus \Lambda$, then

$$\begin{aligned} x_1 \cdots x_{q-1}[(x_\lambda \mid \lambda \in \Lambda) \colon x_q] &= x_1 \cdots x_{q-1}[(x_\lambda \mid \lambda \in \Lambda) \colon x_k] \\ &\subseteq (x_\lambda^2 \mid \lambda \in \Lambda). \end{aligned}$$

Thus we have

$$(x_1t) \cdots (x_{q-1}t)\overline{at^{n+m(q-1)}} \in ((x_1t)^{m+1}, \ldots, (x_{q-1}t)^{m+1})G(\mathfrak{q}).$$

This implies that

$$\begin{aligned} &\frac{G(\mathfrak{q})}{((x_1t)^m, \ldots, (x_{q-1}t)^m)G(\mathfrak{q})}(m(q-1)) \\ &\qquad \xrightarrow{(x_1t)\cdots(x_qt)} \frac{G(\mathfrak{q})}{((x_1t)^{m+1}, \ldots, (x_{q-1}t)^{m+1})G(\mathfrak{q})}((m+1)(q-1)) \end{aligned}$$

is a zero map in degree n. Taking direct limit, we have $[H^{q-1}_{(x_1t,\ldots,x_qt)}(G(\mathfrak{q}))]_n = 0$ for $n < -q+1$. □

Theorem A.4 ([15, Proposition 4.9])**.** *For $1 \leq p \leq s$,*

(A.4.1) $$[H^p_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))]_n = 0 \quad \textit{if } n < 2-p \textit{ or } n \geq 0.$$

Furthermore

(A.4.2) $$H^0_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})) = [H^0_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))]_0 = 0 \colon {}_A x_1.$$

Proof. We work by induction on s. Let $B = A/0 \colon {}_A x_1$. Then $x_1, \ldots, x_s$ is also a d^+-sequence on B and x_1 is B-regular. Furthermore,

$$0 \to 0 \colon {}_A x_1 \to R(\mathfrak{q}) \to R(\mathfrak{q}B) \to 0$$

is exact because $0 \colon {}_A x_1 \cap \mathfrak{q}^n = 0$ if $n > 0$. Therefore, $H^0_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))$ is concentrated in degree 0 and $[H^0_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))]_0 = 0 \colon {}_A x_1$. Thus we obtain (A.4.2). Furthermore, if $p > 0$, then $H^p_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})) \cong H^p_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}B))$. We may assume that $0 \colon {}_A x_1 = 0$ to prove (A.4.1).

If $s = 1$, then $R(\mathfrak{q}) \cong A[T]$ where T is an indeterminate. Since x_1, T is a regular sequence on $A[T]$, $H^1_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})) = 0$.

Next we assume that $s > 1$. There exist the following four short exact sequences

$$0 \to R(\mathfrak{q}) \xrightarrow{x_1} R(\mathfrak{q}) \to R(\mathfrak{q})/x_1R(\mathfrak{q}) \to 0,$$

$$0 \to G(\mathfrak{q})(-1) \xrightarrow{x_1t} R(\mathfrak{q})/x_1R(\mathfrak{q}) \to R(\mathfrak{q}/x_1A) \to 0,$$

$$0 \to R(\mathfrak{q})(-1) \xrightarrow{x_1t} R(\mathfrak{q}) \to R(\mathfrak{q})/x_1tR(\mathfrak{q}) \to 0,$$

$$0 \to A \xrightarrow{x_1} R(\mathfrak{q})/x_1tR(\mathfrak{q}) \to R(\mathfrak{q}/x_1A) \to 0$$

because $(x_1) \cap \mathfrak{q}^n = x_1\mathfrak{q}^{n-1}$. Taking local cohomology, we obtain exact sequences

$$H^{p-1}_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})/x_1R(\mathfrak{q})) \to H^p_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})) \xrightarrow{x_1} H^p_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})), \tag{A.4.3}$$

$$H^{p-1}_{\mathfrak{q}t}(G(\mathfrak{q}))(-1) \to H^{p-1}_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})/x_1R(\mathfrak{q})) \to H^{p-1}_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}/x_1A)), \tag{A.4.4}$$

$$H^{p-1}_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})/x_1tR(\mathfrak{q})) \to H^p_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))(-1) \xrightarrow{x_1t} H^p_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})), \tag{A.4.5}$$

$$H^{p-1}_{\mathfrak{q}}(A) \to H^{p-1}_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})/x_1tR(\mathfrak{q})) \to H^{p-1}_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}/x_1A)). \tag{A.4.6}$$

For $2 \le p \le s$, we have

$$[H^{p-1}_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}/x_1A))]_n = 0 \quad \text{if } n < 2-p+1 \text{ or } n \ge 0$$

and

$$[H^{p-1}_{\mathfrak{q}t}(G(\mathfrak{q}))(-1)]_n = 0 \quad \text{if } n \ne 2-p$$

by the induction hypothesis and Theorem A.3. Therefore, if $n < 2-p$ or $n \ge 0$, then we have

$$[H^{p-1}_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})/x_1R(\mathfrak{q}))]_n = 0$$

by using (A.4.4). Hence we find that

$$[H^p_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))]_n \xrightarrow{x_1} [H^p_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))]_n$$

is a monomorphism by using (A.4.3). This implies $[H^p_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))]_n = 0$ because $H^p_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))$ is annihilated by some power of x_1 in elementwise.

Next we consider $H^1_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))$. Since $H^0_{(\mathfrak{q}t)}(G(\mathfrak{q})) \cong H^0_{\mathfrak{q}}(A) = 0 :_A x_1 = 0$, we have

$$[H^0_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})/x_1R(\mathfrak{q}))]_n = [H^0_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})/x_1tR(\mathfrak{q}))]_n = 0 \quad \text{if } n \ne 0$$

by using (A.4.2), (A.4.4) and (A.4.6). Therefore we find that

$$[H^1_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))]_n \xrightarrow{x_1} [H^1_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))]_n$$

and

$$[H^1_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))]_{n-1} \xrightarrow{x_1t} [H^1_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q}))]_n$$

are monomorphisms for $n \ne 0$ by using (A.4.3) and (A.4.5). Hence $H^1_{(\mathfrak{q},\mathfrak{q}t)}(R(\mathfrak{q})) = 0$ because it is annihilated by some power of x_1 and x_1t in elementwise. □

The following theorem comes from Theorem A.4 and [14, Theorem 3.1.1] at once.

Theorem A.5. *Let r be an integer such that $r \ge s-1$. Then*

$$H^p_{(\mathfrak{q},\mathfrak{q}^rt)}(R(\mathfrak{q}^r)) = 0 \quad \textit{for } 1 \le p \le s.$$

Furthermore

$$H^0_{(\mathfrak{q},\mathfrak{q}^rt)}(R(\mathfrak{q}^r)) = [H^0_{(\mathfrak{q},\mathfrak{q}^rt)}(R(\mathfrak{q}^r))]_0 = 0 :_A x_1.$$

Appendix B. An Example

We give an example of a Noetherian local ring with large non-Cohen-Macaulay locus and construct its Macaulayfication according to Corollary 4.2. We used [3] to calculate our example.

Example B.1. Let k be a field, A the affine semigroup ring

$$k[a, b, c, d, e^2, e^3, ade, bde, cde, d^2e]$$

and $\mathfrak{m}$ the homogeneous maximal ideal of A. Then $\dim A_\mathfrak{m} = 5$ and the non-Cohen-Macaulay locus of $\operatorname{Spec} A_\mathfrak{m}$ is of dimension 3. The sequence $x_1 = a^4$, $x_2 = b^4$, $x_3 = c^4$, $x_4 = d^4$, $x_5 = e^4$ is a p-standard system of parameters of type 3 for $A_\mathfrak{m}$.

Proof. Since A is a finite algebra over $k[a, b, c, d, e^2]$, $\dim A_\mathfrak{m} = 5$ and a, b, c, d, e^2 is a system of parameters for $A_\mathfrak{m}$.

Next we compute $\mathfrak{a}(A_\mathfrak{m})$. We regard A as a graded ring in natural way. Let

$$C = A/(ad, bd, cd, d^2, e^2, e^3, ade, bde, cde, d^2e).$$

Then there exist two short exact sequences of A-modules:

$$0 \to A \to k[a, b, c, d, e] \to C(-1) \to 0, \tag{B.1.1}$$

$$0 \to A/\mathfrak{m}(-1) \to C \to A/(d, e^2, e^3, ade, bde, cde, d^2e) \to 0. \tag{B.1.2}$$

Since $A/(d, e^2, e^3, ade, bde, cde, d^2e) \cong k[a, b, c]$, we find that

$$H^0_\mathfrak{m}(C) = A/\mathfrak{m}(-1),$$
$$H^3_\mathfrak{m}(C) \cong H^3_{(a,b,c)}(k[a, b, c])$$

and

$$H^p_\mathfrak{m}(C) = 0 \quad \text{if } p \neq 0,\ 3.$$

Hence

$$H^1_\mathfrak{m}(A) \cong A/\mathfrak{m}(-2),$$
$$H^4_\mathfrak{m}(A) \cong H^3_{(a,b,c)}(k[a, b, c])(-1)$$

and

$$H^p_\mathfrak{m}(A) = 0 \quad \text{for } p \neq 1,\ 4,\ 5.$$

Thus we have $\mathfrak{a}(A_\mathfrak{m}) = (d, e^2, e^3, ade, bde, cde, d^2e)\mathfrak{m}A_\mathfrak{m}$, $\dim A_\mathfrak{m}/\mathfrak{a}(A_\mathfrak{m}) = 3$ and d^4, $e^4 \in \mathfrak{a}(A_\mathfrak{m})$.

To compute $\mathfrak{a}(A_\mathfrak{m}/(d^4, e^4)A_\mathfrak{m})$, we take Koszul homologies of (B.1.1) with respect to (d^4, e^4). Since $(d^4, e^4)C = 0$, we obtain an exact sequence

$$0 \to C^2(-5) \to A/(d^4, e^4) \to k[a, b, c, d, e]/(d^4, b^4)k[a, b, c, d, e] \to C(-1) \to 0.$$

Therefore

$$H^0_\mathfrak{m}(A/(d^4, e^4)) = (A/\mathfrak{m})^2(-6),$$
$$H^1_\mathfrak{m}(A/(d^4, e^4)) = A/\mathfrak{m}(-2),$$
$$H^2_\mathfrak{m}(A/(d^4, e^4)) = 0.$$

Thus $\mathfrak{a}(A_\mathfrak{m}/(d^4, e^4)A_\mathfrak{m}) = \mathfrak{m}^2 A_\mathfrak{m} \ni c^4$.

Lemma 2.5 says that

$$(d^4, e^4)A_{\mathfrak{m}} \colon c^4 \subseteq (d^4, e^4)A_{\mathfrak{m}} \colon \mathfrak{m}^2 A_{\mathfrak{m}}$$

and hence $(d^4, e^4)A \colon c^4/(d^4, e^4)A$ has finite length. Therefore we obtain an exact sequence

$$\begin{aligned} 0 \to & \frac{(d^4, e^4)A \colon c^4}{(d^4, e^4)}(-4) \to H^0_{\mathfrak{m}}(A/(d^4, e^4))(-4) \\ & \xrightarrow{c^4} H^0_{\mathfrak{m}}(A/(d^4, e^4)) \to H^0_{\mathfrak{m}}(A/(c^4, d^4, e^4)) \to H^1_{\mathfrak{m}}(A/(d^4, e^4))(-4) \\ & \xrightarrow{c^4} H^1_{\mathfrak{m}}(A/(d^4, e^4)) \to H^1_{\mathfrak{m}}(A/(c^4, d^4, e^4)) \to H^2_{\mathfrak{m}}(A/(d^4, e^4))(-4) \\ & \xrightarrow{c^4} H^2_{\mathfrak{m}}(A/(d^4, e^4)) \to \cdots . \end{aligned}$$

See [11, Proposition 2.6]. By comparing degree, we have two split short exact sequences

$$\begin{aligned} 0 \to H^0_{\mathfrak{m}}(A/(d^4, e^4)) \to H^0_{\mathfrak{m}}(A/(c^4, d^4, e^4)) \to H^1_{\mathfrak{m}}(A/(d^4, e^4))(-4) \to 0, \\ 0 \to H^1_{\mathfrak{m}}(A/(d^4, e^4)) \to H^1_{\mathfrak{m}}(A/(c^4, d^4, e^4)) \to H^2_{\mathfrak{m}}(A/(d^4, e^4))(-4) \to 0. \end{aligned}$$

Therefore

$$\begin{aligned} H^0_{\mathfrak{m}}(A/(c^4, d^4, e^4)) &= (A/\mathfrak{m})^3(-6), \\ H^1_{\mathfrak{m}}(A/(c^4, d^4, e^4)) &= A/\mathfrak{m}(-2) \end{aligned}$$

and hence $\mathfrak{a}(A_{\mathfrak{m}}/(c^4, d^4, e^4)) = \mathfrak{m}^2 A_{\mathfrak{m}} \ni b^4$.

Similarly we have $H^0_{\mathfrak{m}}(A/(b^4, c^4, d^4, e^4)) = (A/\mathfrak{m})^4(-6)$ and

$$\mathfrak{a}(A_{\mathfrak{m}}/(b^4, c^4, d^4, e^4)A_{\mathfrak{m}}) = \mathfrak{m}A_{\mathfrak{m}} \ni a^4. \quad \square$$

Although the finite morphism

$$\operatorname{Spec} k[a, b, c, d, e]_{(a,b,c,d,e)} \to \operatorname{Spec} A_{\mathfrak{m}}$$

is a Macaulayfication, we construct a Macaulayfication of $\operatorname{Spec} A_{\mathfrak{m}}$ by using Corollary 4.2. Let $\mathfrak{b} = \prod_{i=1}^4 (x_i, \ldots, x_5)A_{\mathfrak{m}}$ and $Y = \operatorname{Proj} R(\mathfrak{b})$. From the proof of

Lemma 4.3, we find that Y is covered by 16 affine schemes:

$$
\begin{array}{llll}
(1) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{a^4}{e^4}, \dfrac{b^4}{e^4}, \dfrac{c^4}{e^4}, \dfrac{d^4}{e^4}\right], & (2) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{e^4}{a^4}, \dfrac{b^4}{e^4}, \dfrac{c^4}{e^4}, \dfrac{d^4}{e^4}\right], \\
(3) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{a^4}{b^4}, \dfrac{e^4}{b^4}, \dfrac{c^4}{e^4}, \dfrac{d^4}{e^4}\right], & (4) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{b^4}{a^4}, \dfrac{e^4}{b^4}, \dfrac{c^4}{e^4}, \dfrac{d^4}{e^4}\right], \\
(5) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{a^4}{c^4}, \dfrac{b^4}{c^4}, \dfrac{e^4}{c^4}, \dfrac{d^4}{e^4}\right], & (6) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{c^4}{a^4}, \dfrac{b^4}{c^4}, \dfrac{e^4}{c^4}, \dfrac{d^4}{e^4}\right], \\
(7) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{a^4}{b^4}, \dfrac{c^4}{b^4}, \dfrac{e^4}{c^4}, \dfrac{d^4}{e^4}\right], & (8) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{b^4}{a^4}, \dfrac{c^4}{b^4}, \dfrac{e^4}{c^4}, \dfrac{d^4}{e^4}\right], \\
(9) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{a^4}{d^4}, \dfrac{b^4}{d^4}, \dfrac{c^4}{d^4}, \dfrac{e^4}{d^4}\right], & (10) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{d^4}{a^4}, \dfrac{b^4}{d^4}, \dfrac{c^4}{d^4}, \dfrac{e^4}{d^4}\right], \\
(11) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{a^4}{b^4}, \dfrac{d^4}{b^4}, \dfrac{c^4}{d^4}, \dfrac{e^4}{d^4}\right], & (12) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{b^4}{a^4}, \dfrac{d^4}{b^4}, \dfrac{c^4}{d^4}, \dfrac{e^4}{d^4}\right], \\
(13) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{a^4}{c^4}, \dfrac{b^4}{c^4}, \dfrac{d^4}{c^4}, \dfrac{e^4}{d^4}\right], & (14) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{c^4}{a^4}, \dfrac{b^4}{c^4}, \dfrac{d^4}{c^4}, \dfrac{e^4}{d^4}\right], \\
(15) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{a^4}{b^4}, \dfrac{c^4}{b^4}, \dfrac{d^4}{c^4}, \dfrac{e^4}{d^4}\right], & (16) & \operatorname{Spec} A_{\mathfrak{m}}\left[\dfrac{b^4}{a^4}, \dfrac{c^4}{b^4}, \dfrac{d^4}{c^4}, \dfrac{e^4}{d^4}\right].
\end{array}
$$

We check the Cohen-Macaulay property of them.

For example, we consider (16). By letting $s = a$, $t = b/a$, $u = c/b$, $v = d/c$ and $w = e/d$, we obtain an isomorphism

$$
\begin{aligned}
&A[b^4/a^4, c^4/b^4, d^4/c^4, e^4/d^4] \\
&\qquad \cong k[s, t^4, u^4, v^4, w^4, st, stu, stuv, s^2t^2u^2v^2w^2, s^3t^3u^3v^3w^3, \\
&\qquad\qquad s^3t^2u^2v^2w, s^3t^3u^2v^2w, s^3t^3u^3v^2w, s^3t^3u^3v^3w].
\end{aligned}
$$

The right hand side is also an affine semigroup ring and satisfies the assumption of the following theorem.

Theorem B.2 ([13, Theorem 2.6])**.** *Let k be a field and $B = k[m_1, \ldots, m_r]$ an affine semigroup ring, where m_1, ... , m_r are monomials. Assume that*

1. *m_1, ... , m_d are algebraically independent over k;*
2. *there is a positive integer k such that m_{d+1}^k, ... , $m_r^k \in k[m_1, \ldots, m_d]$.*

Then the following statements are equivalent:

1. *B is Cohen-Macaulay;*
2. *$m_iB\colon m_j = m_iB$ for all $i < j$.*

It is easy to see that $A[b^4/a^4, c^4/b^4, d^4/c^4, e^4/d^4]$ is a Cohen-Macaulay ring by using the theorem above and we can prove that (1)–(15) are Cohen-Macaulay in the same way.

References

[1] Yoichi Aoyama and Shiro Goto, *Some special cases of a conjecture of Sharp*, J. Math. Kyoto Univ. **26** (1986), 613–634. MR **88h:**13013

[2] ———, *A conjecture of Sharp—the case of local rings with* dim non-CM ≤ 1 *or* dim ≤ 5, Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata, Kinokuniya, 1988, pp. 27–34. MR **90b:**13018

[3] Dave Bayer and Michael Stillman, *Macaulay: A system for computation in algebraic geometry and commutative algebra*, 1982–1994, Source and object code available for Unix and Macintosh computers. Contact the authors, or download from `math.harvard.edu` via anonymous ftp.

[4] Markus Brodmann, *Kohomologische Eigenschaften on Audblasungen an Lokal Vollständgen Durchschnitten*, 1980, Habilitationsschrift.

[5] ______, *Two types of birational models*, Comment. Math. Helv. **58** (1983), 388–415. MR **85g:**14061

[6] ______, *A few remarks on blowing-up and connectedness*, J. Reine Ang199ew. Math. **370** (1986), 52–60. MR **87j:**14014

[7] Nguyen Tu Cuong, *P-standard systems of parameters and p-standard ideals in local rings*, Acta Math. Vietnamica **20** (1995), 145–161.

[8] Gerd Faltings, *Über die Annulatoren lokaler Kohomogiegruppen*, Arch. Math. (Basel) **30** (1978), 473–476. MR **58:**22058

[9] ______, *Über Macaualyfizierung*, Math. Ann. **238** (1978), 175–192. MR **80c:**14030

[10] Shiro Goto, *On the Cohen-Macaulayfication of certain Buchsbaum rings*, Nagoya Math. J. **80** (1980), 107–116. MR **81m:**13023

[11] ______, *Blowing-up of Buchsbaum rings*, Proceedings, Durham symposium on Commutative Algebra, London Math. Soc. Lect. Notes, vol. 72, Cambridge Univ. Press, 1982, pp. 140–162. MR **84h:**13032

[12] Shiro Goto and Yasuhiro Shimoda, *On Rees algebras over Buchsbaum rings*, J. Math. Kyoto Univ. **20** (1980), 691–708. MR **82c:**13027

[13] Shiro Goto, Naoyoshi Suzuki, and Kei-ichi Watanabe, *On affine semigroup rings*, Japan. J. Math. (N.S.) **2** (1976), 1–12. MR **56:**8553

[14] Shiro Goto and Kei-ichi Watanabe, *On graded rings, I*, J. Math. Soc. Japan **30** (1978), 179–213. MR **81m:**13021

[15] Shiro Goto and Kikumichi Yamagishi, *The theory of unconditioned strong d-sequences and modules of finite local cohomology*, preprint.

[16] A. Grothendieck, *Éléments de Géométrie Algébrique II*, Inst. Hautes Études Sci. Publ. Math., vol. 8, Presses Universitaires de France, 1961. MR **29:**1208

[17] Robin Hartshorne, *Residue and duality*, Lecture Notes in Math., vol. 20, Springer-Verlag, Berlin, Heidelberg, New York, 1966. MR **36:**5145

[18] ______, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer, Berlin, Heidelberg, New-York, 1977. MR **57:**3116

[19] Heisuke Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic 0*, Ann. of Math. (2) **79** (1964), 109–326. MR **33:**7333

[20] Craig Huneke, *The theory of d-sequences and powers of ideals*, Adv. in Math. **46** (1982), 249–279. MR **84g:**13021

[21] Takesi Kawasaki, *On Macaulayfication of certain quasi-projective schemes*, J. Math. Soc. Japan **50** (1998), 969-991. CMP 99:01

[22] Hideyuki Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Math., vol. 8, Cambridge University Press, 1986. MR **90i:**13001

[23] D. G. Northcott and D. Rees, *Reductions of ideals in local rings*, Proc. Cambridge. Phil. Soc. **50** (1954), 145–158. MR **15:**596a

[24] Tetsushi Ogoma, *Existence of dualizing complexes*, J. Math. Kyoto Univ. **24** (1984), 27–48. MR **85j:**13028

[25] Idun Reiten, *The converse to a theorem of Sharp in Gorenstein modules*, Proc. Amer. Math. Soc. **32** (1972), 417–420. MR **45:**5128

[26] Peter Schenzel, *Dualizing complexes and system of parameters*, J. Algebra **58** (1979), 495–501. MR **80f:**14025

[27] ______, *Cohomological annihilators*, Math. Proc. Cambridge Philos. Soc. **91** (1982), 345–350. MR **83j:**13008

[28] ______, *Dualisierende Komplexe in der lokalen Algebra und Buchsbaum ringe*, Lecture Notes in Math., vol. 907, Springer, Berlin, Heidelberg, New York, 1982. MR **83i:**13013

[29] ______, *Standard system of parameters and their blowing-up rings*, J. Reine Angew. Math. **344** (1983), 201–220. MR **84m:**13025

[30] Rodney Y. Sharp, *Necessary conditions for the existence of dualizing complexes in commutative algebra*, Sém. Algèbre P. Dubreil 1977/78, Lecture Notes in Mathematics, vol. 740, Springer-Verlag, 1979, pp. 213–229. MR **81d:**13013

[31] Giuseppe Valla, *Certain graded algebras are always Cohen-Macaulay*, J. Algebra **42** (1976), 537–548. MR **54:**10240

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